A Model of Underwriting and Post-Loss Test without Commitment in Competitive Insurance Market

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Abstract In this article, we establish a model of competitive insurance markets based on Rothschild and Stiglitz (1976) where insurers can perform risk classification tests either before insurance contracts are issued (underwriting) or when coverage claims are filed (post-loss test). However, insurers cannot pre-commit to performing either test in the insurance application period since the tests are costly type-verifications. We derive the perfect Bayesian equilibrium of four cases: no test is used; only one kind of the two types of test is performed; and both tests are performed. The space of parameters where the equilibrium exists in Rothschild and Stiglitz (1976) and Picard (2009) models is extended in our model. The key tradeoff determining which test is utilized lies in the relative magnitude of testing costs. Furthermore, we characterize the contracts provided in the market. Different from the overinsurance counterpart in Picard (2009), the contract for low-risk type with only underwriting test may be either overinsurance, full insurance, or underinsurance in our model.

JEL Classification C72, D81, D82, G22
1 Introduction

As a key component in contract theory, adverse selection has received significant attention both theoretically and empirically. Since the seminal work by Rothschild and Stiglitz (1976), a large body of literature has been presented with focus on various aspects of how insurance mechanisms deal with adverse selection and risk classification.

Generally, there are two kinds of screening mechanisms in the insurance market. The first one is risk classification based on prior information and is termed underwriting since the screening relies on the insurers’ effort to categorize risk types before insurance policies are issued. The second screening mechanism is a test after an insurance claim is filed and thus is called post-loss test. Post-loss test is needed to ensure compliance with the principle of utmost good faith or *uberrimae fidei*. Because insurers rely on the truthfulness of policyholder’s statements in determining their risk classifications, they have the legal right to void the contract should those statements found to be materially false. Typically insurers discover misstatements post loss, during claims investigations.

In literature, the two kinds of risk-classification mechanisms are studied separately. Risk classification using prior information has received extensive attention (e.g. Hoy (1982), Doherty and Thistle (1996), Hoy and Polborn (2000), Hoy and Lambert (2000), and so on). Browne and Kamiya (2012), for instance, theoretically show that an equilibrium (either in Nash type or Wilson type) may exist where insurers offer full coverage even to low-risk type by virtue of committing to an underwriting test instead of simply relying on self-selection.\(^1\) The original work on post-loss tests by Dixit (2000) discusses how costly investigation of risk type after the claim (post-loss auditing) affects the Rothschild and Stiglitz (1976) separating equilibrium.\(^2\) Modifying the model by Dixit (2000), Dixit and Picard (2003) relax the assumption that individuals do not have perfect private information about their risk type. As an important extension, Picard (2009) allows insurers not to commit to the costly verification. Under certain conditions, Picard (2009) shows that a semi-separating equilibrium in the sense of perfect Bayesian equilibrium exists in the insurance market when insurers cannot commit to post-loss testing. Lando (2016) extends Dixit (2000) by investigating the degree to which legal constraints of misrepresentation should be enforced by insurers due to cost-benefit tradeoff. Lando’s perspective is primarily legal rather than economic.

The purpose of this paper is to merge the two lines of inquiry by incorporating underwriting and

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\(^1\)Hence, an insurer’s effort at gathering private information, together with successful testing, may make the first-best outcome possible in Browne and Kamiya (2012).

\(^2\)He shows that the insurer’s ability of post-loss auditing could substantially improve the chance of attaining separating equilibrium and find that *uberrimae fidei* achieves a Pareto improvement compared with the allocation in Rothschild and Stiglitz (1976).
post-loss testing without commitment in one unified framework and to investigate the effects of both tests on the equilibrium in a competitive insurance market. We establish our model on the familiar economy described in Rothschild and Stiglitz (1976), where individual risk type is private information and each insurer is able to supply at least one kind of insurance contract in a competitive market. In our model, the insurers design insurance contracts for high-risk type and low-risk type individuals. An underwriting test is performed before issuance of insurance contracts on individuals who apply for the contract designed for low-risk type. If a high-risk type individual is found to be misrepresenting his or her type in underwriting, the application is cancelled and he or she is faced with a fee charged by the insurer so as to make up the cost of underwriting. After a loss, the individuals claim coverage from their insurers. The insurers can conduct post-loss tests on the insureds who file claims of contract of low-risk type. If a high-risk type individual is found to have misrepresented his or her type in the post-loss test, the insurance premium is refunded and the insurance coverage is voided; a potential fine may be charged as in Picard (2009). We assume that both underwriting and post-loss tests are costly type-verifications, and that the insurers cannot commit to either test. Thus, our model embeds a simplified version of Picard (2009)\(^4\), and extends Browne and Kamiya (2012) into underwriting without commitment. Furthermore, we model underwriting and post-loss tests simultaneously and show the effects of both tests on equilibrium. Intuitively, through underwriting, the insureds’ risk types may become common knowledge and the proportion of the insureds who need to be screened in the post-loss test may be reduced. As a result, the insurers’ strategy on post-loss testing will be affected by underwriting. In an extreme, post-loss tests will be crowded out by underwriting and vice versa. Therefore, it is important to examine the interesting interaction between underwriting and post-loss test, especially when commitment is not made to either.

Since the tests in our model are not committed to by the insurers, we model our economy in a framework of a perfect Bayesian game. In the perfect Bayesian equilibrium, every insurer makes at least non-negative profit while there is no other contracts that, if provided, could generate positive profit and improve the utility of at least one risk type. The equilibrium is actually a second-best Pareto equilibrium\(^5\). We show that if the equilibrium exists, there are four cases of equilibria depending on

\(^3\)In reality, the insurers can decide not to accept the risk and thus deny insurance after underwriting. And in our model, cancelation is without loss of generality since it can be replaced by some exogenous contract. See footnote 8 for details.

\(^4\)The simplification mainly lies in that we consider symmetry in strategies of both the insurers and the insureds. That is, in the equilibrium, all insurers designs the same contract for each type and that the individuals are evenly distributed among insurers in the market. Hence, the insurers cannot infer form the applications the risk types of the applicants. Picard (2009) assumes a finite number of insurers who serve heterogeneous samples of insureds.

\(^5\)The term second-best refers to the fact that adverse-selection cannot be eliminated unless the cost of either test is null.
the fraction of high-risk type individuals in the market and the pair of cost parameters. Conditional on both tests being of sufficiently high cost, neither test will be used by the insurers since they are prohibitively expensive. In this case, the equilibrium depends on the fraction of high-risk type individuals in the market. As in Rothschild and Stiglitz (1976) and Picard (2009), if the fraction of high-risk type individuals in the market is sufficiently low, an equilibrium may not exist. While if the fraction of high-risk type individuals in the market is sufficiently large, and if the costs of both tests are high, the equilibrium coincides with the separating equilibrium in Rothschild and Stiglitz (1976) where the high-risk type individuals receive an actuarially fairly priced full insurance contract while the low-risk type individuals receive a partial insurance contract. As long as either cost of the tests is sufficiently low, an equilibrium always exists. In this case, the test with relatively low testing cost will be performed with positive probability but less than one to make the high-risk type randomize between the two types of contracts. The equilibrium, therefore, is semi-separating equilibrium. In particular, when the cost of the post-loss test is relatively low, the equilibrium allocation coincides with that in Picard (2009) where the contract for high-risk type is an actuarially fairly priced full insurance contract while that for low-risk type individuals provides overinsurance. When the cost of underwriting is relatively low, we establish a new semi-separating equilibrium where the contract for high-risk type is an actuarially fairly priced full insurance contract and that for low-risk type may be either overinsurance, full insurance, or underinsurance, depending on the relationship between the fee collected from misrepresenting individuals and the underwriting cost. Thus, we partially solve the puzzle of overinsurance contract with only post-loss test in Picard (2009), since overinsurance disagrees with the principal of indemnity in insurance practice. Importantly, we derive necessary and sufficient conditions under which both underwriting and post-loss tests are used in the market equilibrium. The resulting contract for low-risk type in this case provides overinsurance and that for high-risk type remains the actuarially fairly priced full insurance contract. In summary, we extend the parameter space in which the equilibrium exists in Rothschild and Stiglitz insurance market and thus contributes to literature.

The remainder of this article is organized as follows. In Section 2, we present our model economy in details. Section 3 solves the Perfect Bayesian Equilibrium. A summary of our findings, as well as concluding remarks, is contained in Section 4.
2 Model Environment

Consider the market environment originally described in Rothschild and Stiglitz (1976) where risk-neutral insurers operate in a competitive market\(^6\) and a continuum of individuals with a standard risk-averse utility function \(u\) such that \(u' > 0\) and \(u'' < 0\) lives in a world where accidents may occur. Individuals are homogeneous in their endowment. Each individual has an initial endowment of \(W_N\). When no accident happens, the individual’s wealth remains \(W_N\). When the accident happens, the individual suffers from a loss of \(A\) and culminates in a state of wealth \(W_A = W_N - A\). The only difference between the individuals is their likelihood of accident. There are two types of risk likelihoods: for a high-risk type (h-type for short) individual, the accident occurs with probability \(\pi_h\), while for a low-risk type (l-type for short) individual, the accident occurs with probability \(\pi_l\); \(0 < \pi_l < \pi_h < 1\). Individuals know their risk type but insurers do not. Both parties know the probability of loss and the proportion of h-type individuals \(\lambda\) and that of l-type individuals \(1 - \lambda\) in the market. An insurance contract \(C\) contains two dimensions denoted by \((k, x)\), where \(k\) is the insurance premium and \(x\) is indemnity less premium.

We assume that both underwriting and the post-loss test are accurate but costly risk verifications.\(^7\) Cost for each underwriting is \(c_1\) and that for each post-loss test is \(c_2\). If the tests are of positive cost, no insurer can pre-commit to them. On one hand, if the high-risk type individuals conjecture that underwriting (post-loss test) is performed on every applicant (policyholder who files claim) of the contract intended for low-risk type, they will self-select the separating contract of high-risk type to avoid insurance contract cancellation. On the other hand, if the insurers are aware of the game theory prediction that the insureds’ choices lead to a separating equilibrium, neither test will be conducted in order to save cost. Therefore, the assumption of underwriting and post-loss testing without commitment is reasonable and realistic.

The game is played sequentially in the following steps:

**Step 1** Nature assigns risk type to each individual. With probability \(\lambda\), an individual is h-type, and with probability \(1 - \lambda\), an individual is l-type. Only the individuals know their risk type.

**Step 2** The insurers design insurance in the competitive market. In particular, they design a contract \(C_h = (k_h, x_h)\) for h-type individuals and a conditional contract \(C_l = (k_l, x_l)\) with underwriting

\(^6\)The competitive market means that it is free for insurers to enter or exit the market and in the equilibrium, no insurer is able to extract positive profit.

\(^7\)An alternative way to modeling probabilistic tests is to assume that the tests are not accurate, and the accuracy of the tests is linked with the testing costs by some one-to-one mapping, just like the setting in Browne and Kamiya (2012). The choice variable then would become the costs of the tests instead of the probabilities of the tests. This alternative setting will not essentially change the model implications.
test which is oriented towards l-type individuals. The conditional contract $C_l$ stipulates that, for any applicant of contract $C_l$, the insurers decide whether to perform the underwriting test. If the underwriting test is performed, the insurers may adjust the contract based on the result of the test: if the applicant is h-type, the insurers cancel the contract for the misrepresenting individual\(^8\) and charge a fee $G \geq 0$ in addition\(^9\), and if the applicant belongs to l-type, the contract remains $C_l$. Once underwriting is performed, the type of the individual being tested becomes common knowledge. And the misrepresenting individuals whose contracts are cancelled cannot apply for other contracts.

**Step 3** Being aware of the details of the contracts, individuals decide whether to purchase insurance, and if they are going to purchase, which one they will choose. Each individual can apply for only one contract, but can randomize between the contracts. That is, the h-type individuals may apply for $C_l$ with probability $\sigma_{hl}$ and for $C_h$ with probability $1 - \sigma_{hl}$. Similarly, the l-type individuals may apply for $C_l$ with probability $\sigma_{ll}$ and for $C_h$ with probability $1 - \sigma_{ll}$.

**Step 4** Based on applications, the insurers decide their underwriting strategy and adjust conditional contracts.\(^{10}\) Specifically, insures may perform underwriting with probability $p_1$. Premiums or fees are paid after underwriting.

**Step 5** Nature decides whether an accident happens for an individual. For h-type, the accident happens with probability $\pi_h$, and for l-type, the accident happens with probability $\pi_l$. Those who suffer from accidents claim coverage from their insurers.

**Step 6** Based on the claims, the insurers decide whether to perform post-loss test for policy holders of $C_l$ who are not tested in underwriting. In particular, the post-loss test can be performed with probability $p_2$. Based on the result of the post-loss test, the insurers pay indemnity in case of l-type or cancel the contract in case of h-type. If the contract is canceled, the premium is refunded by the insurer, and the misrepresenting individual suffers from an additional fine $F \geq 0$, collected by a third party like the government.

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\(^8\)Here, the assumption of cancellation of the application in case of a misrepresenting individual in underwriting can be substituted by exogenously rendering that individual with an insurance contract that breaks even by itself. This contract plus a fee charged must deliver the misrepresenting individual expected utility less than that derived by contract $C_h$, since appropriate penalty is necessary in underwriting in our model. For simplicity, we assume no contract updated here. Endogenizing an updated contract for the misrepresenting individual in underwriting causes potential difficulty and is discussed in the concluding section.

\(^9\)We model the underwriting fee by referring to Browne and Kamiya (2012).

\(^{10}\)Note that we have assumed that once underwriting is performed, the type of the individual being tested becomes common knowledge in Step 2. However, this may rise the issue of free-riding the underwriting test for the insurers. To simplify the problem, we make the insurers commit to their underwriting strategies so as to preclude free-riding.
We denote by $U_l(C)$ the expected utility of $l$-type derived from contract $C$ before underwriting and by $U_h(C)$ the expected utility of $h$-type derived from contract $C$ before underwriting respectively.

3 The Perfect Bayesian Equilibrium

The perfect Bayesian equilibrium of the above game is defined by achieving four standards: First, given the insurers’ contracts, the individuals choose insurance demand $\sigma_{hl}$ and $\sigma_{ll}$ to maximize their expected utility. Second, the insurers’ testing strategies $p_1$ and $p_2$ must maximize expected profits (or equivalently minimize expected payments in case of post-loss test). Third, the contracts $C_h$ and $C_l$ generate at least non-negative profits for the insurers who provide the corresponding contract. Lastly, the insurers form beliefs about the proportion of $h$-type individuals who apply for contract $C_l$ in Step 4, denoted by $\mu_1$, and that of the proportion of $h$-type individuals who claim for indemnity but are not tested in underwriting in Step 6, denoted by $\mu_2$.\(^{11}\) Beliefs should be consistent with the choices of the individuals.

If there is no other equilibrium allocation with non-negative profit that can improve the utility of either the $h$-type or the $l$-type, we say that the perfect Bayesian equilibrium is also second-best Pareto-optimal. In addition, with little loss of generality, we consider a symmetric equilibrium which means that in the equilibrium insurers supply the same contract for a given type and that the insureds distribute evenly among all the insurers. In this case, no insurer can infer from the sample of customers on the risk type of any individual in that sample. In this paper, we consider the second-best Pareto-optimal perfect Bayesian equilibrium that is also symmetric and simply term it as equilibrium.

The following lemmas can be directly obtained from the environment.

**Lemma 1.** There is no pooling equilibrium.

**Proof.** Firstly, underwriting would not be utilized in a pooling equilibrium should it exist to save testing cost and improve the utility of at least one risk type.\(^{12}\) Then Dixit (2000) and Picard (2009) has demonstrated that there is no pooling equilibrium without underwriting. \(\square\)

As shown in Rothschild and Stiglitz (1976), although a pooling contract cannot form an equilibrium in Nash sense, it can upset other potential equilibria if it provides higher utility for both types. Next,  

\(^{11}\)The insurers also need to form belief of the proportion of the $l$-type individuals show apply for $C_h$ in Step 4. However, we can show in Lemma 4 that the $l$-type individuals will not pool with $h$-type in contract $C_h$. So, the insurers’ belief of the proportion of the $l$-type individuals who apply for $C_h$ is 0.

\(^{12}\)Although it is tempting to think that if the insurers could charge sufficiently large fee $G$, they would have more incentive to increase probability of underwriting, however, this effect will be offset by the $h$-risk type individuals’ decreasing incentive to apply for contract $C_l$ due to the fear of no insurance plus a fee. Thus, should a pooling contract emerges in an equilibrium, underwriting would not be used.
we assume that the equilibrium exists without a pooling contract, and introduce the pooling contract when identifying the equilibrium after we solve the contracts in different cases.

**Lemma 2.** The utility for h-type individuals must be at least $U_h(C^*_h)$ in the equilibrium, where $C^*_h$ is the separating contract in Rothschild and Stiglitz (1976); see Figure 1.

*Proof.* Since the contract $C^*_h$ breaks even, any insurer is able to supply contract $C^*_h$ to attract h-type. So the least utility for h-type cannot be lower than $U_h(C^*_h)$. \qed

**Lemma 3.** There cannot be cross-subsidies between contracts $C_h$ and $C_l$ in the equilibrium.

*Proof.* Because the insurers operate in a competitive market, if there is cross-subsidy between contracts $C_h$ and $C_l$, then any insurer as a new entrant may only choose to provide one contract that generates positive profit, upsetting the original equilibrium in Step 2. \qed
Based on the above deduction, the profit for contract $C_h$ is:

$$\Pi_h(C_h) = (1 - \pi_h)k_h - \pi_h x_h,$$

which means expected income from premium less expected claim payment. The profit from l-type for contract $C_l$ is:

$$\Pi_l(C_l) = (1 - \pi_l)(p_1(-c_1 + k_l) + (1 - p_1)k_l) - \pi_l(p_1(c_1 + x_l) + (1 - p_1)(p_2c_2 + x_l))$$

$$= (1 - \pi_l)k_l - \pi_l x_l - p_1 c_1 - (1 - p_1)\pi_l p_2 c_2,$$

where additional cost of tests are deducted. And the profit from h-type for contract $C_l$ is:

$$\Pi_h(C_l) = (1 - \pi_h)(p_1(G - c_1) + (1 - p_1)k_l) - \pi_h(p_1(c_1 - G) + (1 - p_1)(p_2c_2 + (1 - p_2)x_l))$$

$$= (1 - p_1)((1 - \pi_h)k_l - \pi_h(p_2c_2 + (1 - p_2)x_l)) - p_1 c_1 - G,$$

wherein if the insurers do not perform underwriting, their profit is $(1 - \pi_h)k_l - \pi_h(p_2c_2 + (1 - p_2)x_l)$, and when they perform underwriting, they have to pay cost $c_1$ but are able to collect fee $G$ from the misrepresenting h-type individual.

According to Bayesian rule, the insurers’ belief about the proportion of of the h-type individuals who are not tested in underwriting and claim for indemnity in Step 6 is:

$$\mu_2 = \frac{\lambda \pi_h(1 - p_1)\sigma_{hl}}{\lambda \pi_h(1 - p_1)\sigma_{hl} + (1 - \lambda)\pi_l(1 - p_1)\sigma_{ll}}$$

$$= \frac{\lambda \pi_h \sigma_{hl}}{\lambda \pi_h \sigma_{hl} + (1 - \lambda)\pi_l}.$$

The belief is irrelevant to the insurers’ underwriting strategy due to the assumption that once underwriting is performed the type of the individual being tested becomes common knowledge. Then the expected payment of the post-loss test is:

$$b_2 = (1 - p_2)x_l + p_2(c_2 + 0 + (1 - \mu_2)x_l)$$

$$= x_l + p_2(c_2 - \mu_2 x_l).$$

Lemma 1 in Picard (2009) demonstrates that $p_2$ must be lower than 1 for $c_2 > 0$. Otherwise, all h-type individuals passing themselves off as l-type would be verified and faced with fines. They would be better off choosing contract $C_h$. Then $\sigma_{hl}$ and thus $\mu_2$ will be 0, and for the insurers providing $C_l$, it is optimal to choose $p_2 = 0$ to minimize the payment of the post-loss test. Thus, when the post-loss test is performed with positive probability, the expected payment of no testing must be equal to that of testing, that is:

$$x_l = c_2 + (1 - \mu_2)x_l,$$
or equivalently
\[ c_2 = \mu_2 x_1. \]  
(7)

Based on the reasoning above, we always have:
\[ b_2 = x_1, \]  
(8)

which means that in the optimal post-loss testing strategy, the expected payment of post-loss test is always equal to the insurance indemnity less premium.

The insurers’ belief about the proportion of h-type individuals who apply for \( C_l \) in Step 4 is:
\[ \mu_1 = \frac{\lambda \sigma_h l}{\lambda \sigma_h l + (1 - \lambda)}. \]  
(9)

Then the expected profit\(^{13}\) of the underwriting is:
\[
\begin{align*}
    b_1 &= (1 - p_1) \left( (1 - (\mu_1 \pi_h + (1 - \mu_1) \pi_l)) k_l - (\mu_1 \pi_h + (1 - \mu_1) \pi_l) b_2 \right) \\
    &\quad + p_1 (\mu_1 (G - c_1) + (1 - \mu_1) ((1 - \pi_l) k_l - \pi_l x_l - c_1)) \\
    &= (1 - (\mu_1 \pi_h + (1 - \mu_1) \pi_l)) k_l - (\mu_1 \pi_h + (1 - \mu_1) \pi_l) x_l \\
    &\quad - p_1 (c_1 - \mu_1 G + \mu_1 ((1 - \pi_h) k_l - \pi_h x_l)),
\end{align*}
\]
(10)

where \( (1 - (\mu_1 \pi_h + (1 - \mu_1) \pi_l)) k_l - (\mu_1 \pi_h + (1 - \mu_1) \pi_l) x_l \) is the expected profit of contract without underwriting or either test, and \( p_1 (c_1 - \mu_1 G + \mu_1 ((1 - \pi_h) k_l - \pi_h x_l)) \) stands for the net cost of underwriting: for each underwriting test, cost \( c_1 \) is paid and profit \( \mu_1 ((1 - \pi_h) k_l - \pi_h x_l) \) cannot be collected due to the cancellation of the misrepresenting individual’s contract. However, expected fee \( \mu_1 G \) can be recovered in case of cancellation. Clearly, if underwriting is entirely free, i.e. \( c_1 = 0 \), then insurers will perform underwriting with probability 1. In this case, all h-type individuals will only apply for contract \( C_h \) and the first-best contract \( C_l^* \) (see Figure 1) could be offered in the market to attract all l-type individuals, resulting in a first-best allocation. However, similar to the post-loss test, the following lemma shows that the underwriting will not be performed almost surely if the cost of underwriting is positive.

**Lemma 6.** If \( c_1 > 0 \), then \( p_1 < 1 \) in the equilibrium.

**Proof.** If \( p_1 = 1 \), then all the h-type individuals misrepresenting themselves as l-type will be revealed and culminate in cancellation of insurance policies. Their expected utility of no insurance plus a fee

\[^{13}\]In the post-loss test, the income from premium has been determined in Step 4 and the decision whether to perform post-loss test cannot change income part. Thus, the optimal strategy of post-loss test is to minimize the expected payment of post-loss test. For underwriting, on the other hand, since it is performed ex ante and finding an misrepresenting individual results in cancellation of the contract, the strategy of underwriting affects income from premium. So, we have to consider expected profit.
$G$ is lower than $U_h(C^*_h)$. So, h-type individuals will apply for $C_h$ only. Knowing that the equilibrium is separating, the insurers will not perform underwriting since it is costly, a contraction. □

Now without loss of generality, we focus on the case of $c_1 > 0$ and $c_2 > 0$ in the analysis and keep it in mind that in case of either $c_1 = 0$ or $c_2 = 0$, the first-best allocation can be achieved. For underwriting to be performed with positive probability, the profit of no underwriting must equal that of underwriting, namely,

$$
(1 - (\mu_1 \pi_h + (1 - \mu_1) \pi_l))k_l - (\mu_1 \pi_h + (1 - \mu_1) \pi_l)x_l
= -c_1 + \mu_1 G + (1 - \mu_1)(1 - \pi_l)k_l - \pi_l x_l),
$$

or equivalently

$$
c_1 = \mu_1 (G + \pi_h x_l - (1 - \pi_h)k_l).
$$

This means that the condition on which underwriting is performed is that the cost of underwriting is exactly covered by the expected revenue of the test. And for underwriting not to be performed, the profit of underwriting must be no larger than that of no underwriting:

$$
c_1 \geq \mu_1 (G + \pi_h x_l - (1 - \pi_h)k_l).
$$

Since the utility of h-type individuals is fixed by Lemma 5 at $U_h(C^*_h)$, any contract in the equilibrium must maximize the utility of l-type to preclude their deviation. Thus, solving the second-best Pareto-optimal equilibrium is equivalent to solving the following problem:

$$
\max U_l(C_l) = (1 - \pi_l)u(W_N - k_l) + \pi_l u(W_A + x_l)
$$

with respect to $C_l$, $\sigma_{hl}$, and $p_1$, $p_2$, and subject to the following restrictions: The resource constraint:

$$
\lambda((1 - \sigma_{hl})\Pi_h(C_h) + \sigma_{hl}\Pi_h(C_l)) + (1 - \lambda)\Pi_l(C_l) = 0.
$$

The constraint for optimal underwriting: if $p_1 = 0$, then equation (13) must hold, and if $p_1 > 0$, then equation (12) must hold; $\mu_1$ is defined in (9). The constraint for optimal post-loss test: if $p_2 > 0$, then equation (7) must hold, and if $p_2 = 0$, then

$$
\mu_2 x_l \leq c_2,
$$

where $\mu_2$ is defined in (4). And finally the participation constraint for h-type based on Lemma 5:

$$
U_h(C^*_h) = U_h(C_h) = U_h(C_l),
$$

where

$$
U_h(C_h) = (1 - \pi_h)u(W_N - k_h) + \pi_h u(W_A + x_h),
$$

(18)
and
\[ U_h(C_l) = (1 - \pi_h)(p_1 u(W_N - G) + (1 - p_1)u(W_N - k_l)) + \pi_h (p_1 u(W_A - G) + (1 - p_1)(p_2 u(W_A - F) + (1 - p_2)u(W_A + x_l))). \]  

Note that we already have \( C_h = C_h^* \) by Lemma 5, so the resource constraint (15) can be reduced to:
\[ \lambda \sigma_{hl}(C_l) + (1 - \lambda)\Pi_l(C_l) = 0. \]  

4 The Solution to the Equilibrium

The solution to the equilibrium depends on the parameters \( \pi_h, \pi_l, \lambda, c_1, c_2, G, \) and \( F \). In order to distinguish between different cases, we first impose restrictions on the insurers’ choices of the two tests and then identify the equilibrium by reintroducing a pooling contract and comparing value function \( U_l(C_l) \).

4.1 Solutions under Restrictions on the Tests

Clearly, there are four cases depending on whether either, both, or neither test is in use. Two cases wherein underwriting is not used have been discussed in the literature, and we summarize them in the following proposition.

Proposition 1. If the restriction that \( p_1 = p_2 = 0 \) is imposed on problem (14), then the solution is \( C_h = C_h^*, C_l = C_l^{**}, \sigma_{hl} = 0, \) and \( \sigma_{ll} = 1 \).\(^{14}\) If the restrictions that \( p_1 = 0 \) but \( p_2 > 0 \) are imposed on problem (14), then the solution is \( C_h = C_h^*, C_l = \hat{C}_l, \sigma_{hl} = \frac{(1 - \lambda)\pi_l c_2}{\lambda \sigma_{hl}(x_l - c_2)}, \) and \( \sigma_{ll} = 1 \), with \( p_2 \) such that \( U_h(C_h) = U_h(\hat{C}_l) \). The contracts \( C_h^* \) and \( C_l^{**} \) are separating contracts for \( h \)-type and \( l \)-type respectively in Rothschild and Stiglitz (1976). And the contract \( \hat{C}_l \) is the semi-separating contract in Picard (2009) which provides overinsurance if \( c_2 > 0 \).

The remaining two cases involve underwriting with positive probability, \( p_1 > 0 \). We first present a conclusion similar to Lemma 2 in Picard (2009), that is the testing strategy of either underwriting or post-loss test will not enter into the resource constraint.

Lemma 7. When \( p_1 > 0 \), the resource constraint can be reduced into:
\[ \Xi(\sigma_{hl}, k_l, x_l) \equiv \lambda \sigma_{hl}((1 - \pi_h)k_l - \pi_h x_l) + (1 - \lambda)((1 - \pi_l)k_l - \pi_l x_l) = 0. \]  

\(^{14}\)Note that we postpone the discussion that a pooling contract may upset our solutions until the next subsection. So the separating contracts solve the case of \( p_1 = p_2 = 0 \) without depending on \( \lambda \).
Proof. If \( p_1 > 0 \) and \( p_2 = 0 \), then by (9) and (12), the insurers’ belief about the proportion of \( h \)-type individuals applying for contract \( C_l \) is:

\[
\sigma_{hl} = \frac{(1 - \lambda)c_1}{\lambda(\pi_h x_l - (1 - \pi_h)k_l + G - c_1)} \equiv \sigma^U(k_l, x_l). \tag{22}
\]

Substitute (22) into (20), we have (21). If \( p_1 > 0 \) and \( p_2 > 0 \), then by (4) and (7), we have:

\[
\sigma_{hl} = \frac{(1 - \lambda)\pi_l c_2}{\lambda\pi_h(x_l - c_2)} \equiv \sigma^P(k_l, x_l), \tag{23}
\]

in addition to (22). Substitute (22) and (23) into (20), we have (21).

Now consider the case of only using underwriting. The resource constraint (21) combined with (22) is equivalent to:

\[
0 = \pi_h x_l - (\pi_h + \pi_l - 2\pi_h\pi_l)x_lk_l + (1 - \pi_h)(1 - \pi_l)k_l^2
+ c_1(\pi_h - \pi_l)(x_l + k_l) + G(\pi_l x_l - (1 - \pi_l)k_l). \tag{24}
\]

Denote (24) by \( \psi(k_l, x_l) = 0 \). It is easy to show that the above equation represents a hyperbola on the state space.\(^{15}\) To solve (14), we have the first order condition:

\[
-\frac{\partial U_l(C_l)}{\partial k_l} = -\frac{\partial \psi(k_l, x_l)}{\partial k_l}. \tag{25}
\]

Equation (24) together with equation (25), potentially determines a contract, denoted by \( \tilde{C}_l \). However, for the solved contract \( \tilde{C}_l \) to be an interior solution in the restricted problem, \( \sigma_{hl} \) calculated in (22) must be less than 1 to exclude pooling equilibrium with underwriting, which implies that:

\[
\pi_h x_l - (1 - \pi_h)k_l > \frac{c_1}{\lambda} - G. \tag{26}
\]

For \( \lambda \) that is sufficiently small, the contract \( \tilde{C}_l \) cannot satisfy condition (26), resulting in a corner solution. Define:

\[
\tilde{\lambda}(c_1; \pi_h, \pi_l, G) \equiv \sup \left\{ \lambda \in (0, 1) : \pi_h \tilde{x}_l - (1 - \pi_h)\tilde{k}_l \leq \frac{c_1}{\lambda} - G, \right. \]
\[
\left. \text{where} \ (\tilde{k}_l, \tilde{x}_l) \text{ solves system (24) and (25)} \right\}. \tag{27}
\]

Then \( \tilde{\lambda} \) is the maximum \( \lambda \) such that \( \sigma_{hl} = 1 \). As long as \( \lambda > \tilde{\lambda} \), the solution to problem (14) where only underwriting is used is an interior solution so that \( \sigma_{hl} \in (0, 1) \). The problem is solved in the following proposition.

**Proposition 2.** Assume that the restrictions that \( p_1 > 0 \) and \( p_2 = 0 \) are imposed on problem (14). There exists \( \tilde{\lambda}(c_1; \pi_h, \pi_l, G) \) defined in (27), such that when \( \lambda > \tilde{\lambda} \), then \( C_h = C^*_h, C_l = \tilde{C}_l, \sigma_{hl} = 0 \).

\(^{15}\)Note that the discriminant of (24) is equal to \((\pi_h(1 - \pi_l) - \pi_l(1 - \pi_h))^2\) that is larger than 0 given \( \pi_h > \pi_l \).
\[
\sigma^U(\tilde{k}_l, \tilde{x}_l) = \frac{(1-\lambda)c_1}{\lambda(\sigma_h, l - (1-\sigma_h)\tilde{k}_l + G - c_1)} \in (0,1), \text{ with } p_1 \text{ such that } U_h(C_h^*) = U_h(\tilde{C}_l) \text{ solves problem (14)}. 
\]

The contracts \(C_h^*\) is separating contracts for h-type in Rothschild and Stiglitz (1976). And the contract \(\tilde{C}_l\) is solved by system (24) and (25). Besides, if \(c_1 > G\), the contract \(\tilde{C}_l\) provides overinsurance such that \(\tilde{k}_l + \tilde{x}_l > A\). If \(c_1 = G\), the contract \(\tilde{C}_l\) provides full insurance such that \(\tilde{k}_l + \tilde{x}_l = A\). And if \(c_1 < G\), the contract \(\tilde{C}_l\) provides underinsurance such that \(\tilde{k}_l + \tilde{x}_l < A\).

**Proof.** See the Appendix. \(\square\)

Proposition 2 states that when the insurers cannot commit to underwriting, underwriting will be performed such that h-type individuals are indifferent between contract \(C_h^*\) and \(\tilde{C}_l\); hence, h-type individuals will randomize between the two contracts. The Bayesian game requires that the insurers’ belief of the h-type individuals’ demand for contract \(\tilde{C}_l\) be consistent with their strategy. Thus, \(\sigma_{hl}\) is determined by (22). The equilibrium with underwriting, if it exist, will be a semi-separating equilibrium. Different from the over coverage contract \(\tilde{C}_l\) with only post-loss test derived in Picard (2009), the contract with underwriting may take the form of diverse coverage depending on the relationship between the test cost \(c_1\) and fees collected from misrepresenting individuals \(G\).

Intuitively, the fee has two effects: on one hand, given \(\sigma_{hl}\), a large amount of fees charged would relax the budget constraint of insurers who supply contract \(\tilde{C}_l\) and thus may prospectively enable them to provide more coverage. On the other hand, a large amount of fee prohibits h-type individuals from applying for contract \(\tilde{C}_l\), resulting in a lower \(\sigma_{hl}\). Proposition 2 shows that the latter effect dominates and thus, when fee level \(G\) increases, the optimal contract \(\tilde{C}_l\) turns from overinsurance to underinsurance. When \(G = c_1\), the insurers charge exactly the testing cost for the misrepresenting individuals. In this case, the resource constraint (21) is parallel to the fair-odds line of l-type, so contract \(\tilde{C}_l\) provides full insurance. Moreover, we show in the Appendix that no matter how large \(G\) is, as long as \(c_1 > 0\), the resource constraint \(\psi(W_N - W_1, W_2 - W_A) = 0\) is always below the fair-odds line of l-type. Thus, the contract \(\tilde{C}_l\) is second-best. While it is an empirical question on whether in practice the insurers charge more fee than underwriting cost to misrepresenting individuals, we expect that the case of \(G > c_1\) seems to be more realistic in light of the principal of indemnity. Figure 2 plots contract \(\tilde{C}_l\) in different cases in the state space.

![Figure 2 about here](image-url)

From Figure 2, we also observe that when the cost of underwriting approaches 0, the budget constraint \(\psi(k_l, x_l) = 0\) (the dark solid curve) approximates the fair-odds line of l-type EL. The first best allocation can be achieved at the limit of \(c_1 = 0\).

An important question is whether problem (14) can be solved under the restriction that both \(p_1\) and \(p_2\) are positive since in reality, more often than not, both tests are utilized together. Conditional
on performing underwriting, Proposition 1 has solved the contract $\hat{C}_l$ for l-type where post-loss test is used and Proposition 2 has solved the contract $\check{C}_l$ for l-type where underwriting is used. It is natural to speculate that a candidate solution to problem (14) where both underwriting and post-loss test are used must be the one where $\hat{C}_l$ and $\check{C}_l$ coincide. Formally, two conditions should be satisfied if both tests are used.

**Condition 1.** The insurers’ belief on the proportion of $h$-type individuals who apply for contract $C_l$ in underwriting must be consistent with that in post-loss testing in the sense that they are based on the same expected strategy of the $h$-type applicants:

$$\sigma^U(k_l, x_l) = \sigma^P(k_l, x_l).$$

(28)

The first condition is derived from the rationality assumption of the Bayesian game’s participants and states that at the optimal contract $C_l$, the two resource constraints match their values (by either intersection or tangency). The second condition reinforces the first condition by introducing a smooth-pasting condition.

**Condition 2.** For a contract that incorporates either test to be second-best Pareto-optimal, the corresponding resource constraint of either test must be tangent to the l-type individuals’ indifference curve, and they share the same point of tangency, which requires that two resource constraints

$$\Xi(\sigma^U(k_l, x_l), k_l, x_l) = 0 \quad \text{and} \quad \Xi(\sigma^P(k_l, x_l), k_l, x_l) = 0$$

must be tangent to each other at the optimal contract $C_l$.

Should the second condition fail, that is when the two resource constraints intersect at the optimal contract for l-type, the insurers would be able to extract positive profit by deviating to a contract that incorporates only one test and dominates the one that incorporates both tests. Since this deviation can also improve at least l-type, it is feasible. In Figure 3, we plot a case where Condition 2 fails. In this figure, the resource constraint with post-loss test intersects with that with underwriting at the optimal contract with underwriting $\hat{C}_l$. There exists a contract $C_l$ which incorporates only post-loss test that, if provided, can generate positive profit and attract all l-type individuals. This contract thus upsets contract $\check{C}_l$. So, for an equilibrium with both tests to exist, Condition 2 is necessary. The following lemma is a direct implication of the second condition and Proposition 2.

**Lemma 8.** If problem (14) is solved with $p_1 > 0$, $p_2 > 0$ and $c_2 > 0$, then $c_1 > G$.

*Proof.* Because the contract with post-loss test provides overinsurance for the l-type if $c_2 > 0$. For the contract with underwriting to coincide with the one with post-loss test, it must provide overinsurance. Then Proposition 2 implies that this is only the case of $c_1 > G$. 

□
Given general parameters, problem (14) with \( p_1 > 0 \) and \( p_2 > 0 \) cannot be solved due to stringent constraints. Assume that the slope of the indifference curve of the l-type that is tangent to resource constraint \( \Xi(\sigma^U(W_N - W_1, W_2 - W_1), W_N - W_1, W_2 - W_1) = 0 \) is \(-s < \frac{1 - \pi_l}{\pi_l}\). Then Condition 2 means that:

\[
-\Xi_x(\sigma^U, k_l, x_l) \frac{\partial \sigma^U}{\partial k_l} + \Xi_{k_l}(\sigma^U, k_l, x_l) = s = -\Xi_x(\sigma^P, k_l, x_l) \frac{\partial \sigma^P}{\partial k_l} + \Xi_{k_l}(\sigma^P, k_l, x_l),
\]

where \( \Xi_x, \Xi_{k_l}, \) and \( \Xi_{x_l} \) denote the partial derivative of \( \Xi \) with respect to the first, second, and third argument respectively, and that system:

\[
\begin{pmatrix}
\Xi(\sigma^U(k_l, x_l), k_l, x_l) \\
\Xi(\sigma^P(k_l, x_l), k_l, x_l)
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

has one and only untrivial solution, which is equivalent to one equation on parameters\(^{16}\):

\[
((\pi_h - \pi_l)\pi_h c_2 - \pi_h ((\pi_h - \pi_l)c_1 + \pi_l G) + \pi_h G)^2 = 4\pi_h (1 - \pi_h)((\pi_h - \pi_l)c_1 + \pi_l G)c_2. \quad (31)
\]

This equation, denoted by \( \varphi(c_1, c_2) = 0 \), together with (28) and (29), forms a system of four equations. Thus the contract \((k_l, x_l)\) can be solved only with special parameters. Conditional on it is solved, there exists a pair of cost parameters \((\tilde{c}_1', \tilde{c}_2')\) with \(\pi_h, \pi_l, \) and \(G\) being free parameters, that guarantee the existence of the solution to the system.\(^{17}\) Conversely, given

\[
(c_1, c_2) = (c_1(G; \pi_h, \pi_l), c_2(G; \pi_h, \pi_l)) \equiv (\tilde{c}_1', \tilde{c}_2'),
\]

the system of four equations (28), (29), and (31) can be solved by some contract. We identify the contract by finding the sufficient conditions imposed on parameters, in particular the pair \((\tilde{c}_1', \tilde{c}_2')\), under which an optimal contract, denoted by \(\tilde{C}'_l\), that solves (24) and (25), also satisfies Condition 1 and Condition 2 (namely equations (28), (29), and (31)). Formally, we have the following proposition.

**Proposition 3.** Assume that \(\lambda > \bar{\lambda}\). Denote by \(\tilde{C}'_l\) a contract that solves (24) and (25) such that:

\[
s = \frac{\partial U_l(\tilde{C}'_l)/\partial k_l}{\partial U_l(\tilde{C}'_l)/\partial x_l}. \quad (33)
\]

If problem (14) with restrictions \( p_1 > 0 \) and \( p_2 > 0 \) can be solved by \(C_l = \tilde{C}'_l\), then the cost parameters must be determined by \((c_1, c_2) = (\tilde{c}_1', \tilde{c}_2')\) which are solutions to a system\(^{18}\) combing

\[
\varphi(\tilde{c}_1', \tilde{c}_2') = 0, \quad (34)
\]

and

\[
\tilde{c}_2' = \frac{\pi_h s}{\pi_h \pi_l s - (1 - \pi_l)\pi_l \tilde{c}_1'}, \quad (35)
\]

\(^{16}\)See details for this equation in the Appendix.

\(^{17}\)Note that the solution does not explicitly depend on parameters \(F\) and \(\lambda\) if \(\lambda > \bar{\lambda}\).

\(^{18}\)Note that \(s\) depends on \(c_1\) and thus is endogenously determined. So equation (35) in the system is highly nonlinear.
such that \((c'_1, c'_2)\) are above \(\left( G, \frac{1 - \pi_h}{\pi_h - \pi_l} G \right)\) on \(\varphi(c_1, c_2) = 0\). Conversely, if \(\sqrt{1 - \pi_h} > 1 - \pi_l\), \(\frac{1}{(1 - \sqrt{1 - \pi_h})^2} > \frac{1 - \pi_h}{\pi_h - \pi_l} G\), and \(\frac{\pi_h}{\pi_h - (1 - \pi_h)\pi_l} > \frac{1 - \pi_h}{\pi_h - \pi_l}\), then \((c_1, c_2) = (c'_1, c'_2)\) solved by (34) and (35) exist, and problem (14) with probabilities that can provide \(h\)-type individuals with expected utility of \(U\) whose contracts are cancelled cannot apply for other contracts (no renegotiation), any pair of testing probabilities that can support the equilibrium which is still semi-separating. The contracts \(C'_h\) is separating contracts for \(h\)-type in Rothschild and Stiglitz (1976).

The first part of Proposition 3 gives necessary conditions of cost parameters \((c_1, c_2)\) when both underwriting and post-loss tests are used. The second part of Proposition 3 provides sufficient conditions imposed on parameters such that the solution to problem (14) with underwriting solved in Proposition 2 can also allow for post-loss testing and thus both tests are used. It can be seen from Proposition 3 that although underwriting and post-loss testing are commonly used together, their co-existence in the Rothschild and Stiglitz insurance market is highly demanding if the insurers can commit to neither test: only when cost parameters satisfy specific conditions can the problem be solved with both tests. And from the proof of Proposition 2, it can be shown that \(\frac{\partial \tilde{\gamma}}{\partial c_2} > 0\) if \((c'_1, c'_2)\) exist. In Figure 4, we plot an example in which a contract \(\tilde{C}'_l\) solves problem (14) with both tests. Conditional on it is solved, the optimal contract \(\tilde{C}'_l\) is located where the contract with underwriting and the contract with post-loss test coincide and it provides over coverage. The equation \(U_h(C'_h) = U_h(\tilde{C}'_l)\) specifies testing probabilities \(p_1\) and \(p_2\). However, there are infinite numbers of pairs of \((p_1, p_2)\) such that \(U_h(C'_h) = U_h(\tilde{C}'_l)\). Since the insurance contract can be purchased only once at Step 3, and those whose contracts are cancelled cannot apply for other contracts (no renegotiation), any pair of testing probabilities that can provide \(h\)-type individuals with expected utility of \(U_h(C'_h)\) make the \(h\)-type indifferent and so for the insurers who provide the contract due to Lemma 7. Therefore, the solution supports the equilibrium which is still semi-separating.

Figure 4 about here

4.2 Identification of the Equilibrium

Until now, we have solved problem (14) in four cases. When no tests are used, the optimal contract for \(l\)-type is the separating contract \(C^*_{l}\) in Rothschild and Stiglitz (1976). When only post-loss test is used, the optimal contract for \(l\)-type is the semi-separating contract \(\tilde{C}_l\) in Picard (2009). When only underwriting is used, the optimal contract is the semi-separating contract \(\tilde{C}_l\) we solve in Proposition 2. And with special cost parameters, we show that the semi-separating contract \(\tilde{C}_l\) could allow for both underwriting and post-loss test, which we denote by \(\tilde{C}'_l\). Among the four contracts, only one contract can dominate in equilibrium, namely rendering highest utility for \(l\)-type individuals, dependent on the parameters. The key tradeoff lies in the costs of the tests. We define \(g(c_1; \lambda) \equiv U_l(\tilde{C}'_l)\) for \(\lambda > \tilde{\lambda}(c_1)\).
and \( g(c_1; \lambda) = -\infty \) otherwise as the utility derived from contract \( \hat{C}_l \) for the l-type, and \( f(c_2) \equiv U_l(\hat{C}_l) \) as that derived from contract \( \hat{C}_l \) for the l-type. Clearly, \( \frac{dg(c_1)}{dc_1} < 0 \) for \( \lambda > \hat{\lambda}(c_1) \) and \( \frac{df(c_2)}{dc_2} < 0 \).

It is shown in Rothschild and Stiglitz (1976) that if a pooling contract with neither test does not upset the separating contract \( C_l^{**} \), the proportion of the high risk type individuals in the market must be larger than or equal to a cutoff level, denoted by \( \lambda_{RS} \). Suppose first that \( \lambda \geq \lambda_{RS} \). If contract \( \tilde{C}_l \) dominates \( C_l^{**} \), namely \( g(c_1) \geq U_l(C_l^{**}) \), then it must be the case that \( \hat{\lambda}(c_1) \leq \lambda_{RS} \); see Figure 5. Thus \( \lambda > \hat{\lambda}(c_1) \) can be ensured. Define \( c_1^* \) such that \( g(c_1^*) = U_l(C_l^{**}) \). Since we have \( \frac{dg(c_1)}{dc_1} < 0 \), then when \( c_1 \leq c_1^* \), \( \tilde{C}_l \) dominates \( C_l^{**} \) and when \( c_1 > c_1^* \), \( C_l^{**} \) dominates \( \tilde{C}_l \). Picard (2009) has shown that there exists \( c_2^* \) such that \( f(c_2^*) = U_l(C_l^{**}) \). And when \( c_2 \leq c_2^* \), \( \tilde{C}_l \) dominates \( C_l^{**} \), while when \( c_2 > c_2^* \), \( C_l^{**} \) dominates \( \tilde{C}_l \). If we have both \( c_1 \leq c_1^* \) and \( c_2 \leq c_2^* \), then \( g(c_1) = f(c_2) \) defines a function that \( c_2 = f^{-1}(g(c_1)) \) with \( g(0) = f(0) \) and \( g(c_1^*) = f(c_2^*) \). By implicit function theorem, we have \( \frac{dc_2}{dc_1} > 0 \). If \( c_2 < f^{-1}(g(c_1)) \), then \( \tilde{C}_l \) dominates \( \tilde{C}_l \). Otherwise, \( \tilde{C}_l \) dominates \( \tilde{C}_l \).

Figure 5 about here

If \( \lambda < \lambda_{RS} \), \( C_l^{**} \) cannot be a contract for l-type in the equilibrium, since it is dominated by a pooling contract with neither test that is on the market average fair-odds line and maximizes the utility of l-type. For \( \tilde{C}_l \) or \( \tilde{C}_l \) to be the contract in the equilibrium, the pooling contract should be excluded. Denote by \( m(\lambda) \) the utility for l-type individuals if they are provided with the pooling contract without test. If \( \tilde{C}_l \) dominates the pooling contract, we always have \( \hat{\lambda}(c_1) \leq \lambda \). Thus, given \( \lambda < \lambda_{RS} \), \( g(c_1) \geq m(\lambda) \) implies that \( c_1 \leq g^{-1}(m(\lambda)) \). Picard (2009) has shown that for \( \tilde{C}_l \) to dominate the pooling contract, \( c_2 \leq f^{-1}(m(\lambda)) \) must hold. If we have both \( c_1 \leq g^{-1}(m(\lambda)) \) and \( c_2 \leq f^{-1}(m(\lambda)) \), which contract dominates depends on the difference between \( g(c_1) \) and \( f(c_2) \). If \( c_2 < f^{-1}(g(c_1)) \), then \( \tilde{C}_l \) dominates \( \tilde{C}_l \). Otherwise, \( \tilde{C}_l \) dominates \( \tilde{C}_l \). Finally, note that \( m(\lambda_{RS}) = U_l(C_l^{**}) \). Thus, \( c_1^* = g^{-1}(m(\lambda_{RS})) \) and \( c_2^* = f^{-1}(m(\lambda_{RS})) \). Specially, if \( (c_1^*, c_2^*) \in \{(c_1, c_2) : 0 \leq c_1 \leq c_1^*, c_2 = f^{-1}(g(c_1))\} \), we obtain an equilibrium contract where both tests are used.

Figure 6 visualizes the analysis above. Note that in each panel, the red solid curve stands for the pair of cost parameters \( (c_1, c_2) \) at which the contract with only underwriting derives the same utility to l-type as the contract with only post-loss test. As a result, it characterizes the effect of substitution between the two tests. We can observe that underwriting is more expensive than post-loss testing: to be able to deliver the same utility to the l-type individuals, the cost of a compatible underwriting test is lower than that of a post-loss test. This is reasonable since underwriting covers all individuals who apply for contract \( C_l \) while post-loss testing only pertains to policyholders of \( C_l \) who claim for coverage.

Figure 6 about here
We summarize the conclusions in the following proposition.

**Proposition 4.** Given contracts $C^*_h$, $C^{**}_1$, $\tilde{C}_1$, $\check{C}_1$, and $\check{C}'_1$ that are solved in Propositions 1, 2, and 3. If either $c_1 = 0$ or $c_2 = 0$, the equilibrium is a separating equilibrium with the first-best allocation where the test with no cost will be performed with probability 1. When $c_1 > 0$ and $c_2 > 0$: If $c_2 < f^{-1}(g(c_1))$, and $c_2 \leq \min \left\{ f^{-1}(m(\lambda)), c^*_2 \right\}$, the equilibrium is a semi-separating equilibrium with $C_h = C^*_h$, $C_i = \tilde{C}_i$, $\sigma_{hl} = \sigma^P(\tilde{k}_i, \tilde{x}_i)$, $\sigma_{ll} = 1$, $p_1 = 0$, and $p_2$ such that $U_h(C^*_h) = U_h(\check{C}_1)$. If $c_2 \geq f^{-1}(g(c_1))$, and $c_1 \leq \min \left\{ g^{-1}(m(\lambda)), c^*_1 \right\}$, the equilibrium is a semi-separating equilibrium with $C_h = C^*_h$, $C_i = \tilde{C}_i$, $\sigma_{hl} = \sigma^P(\tilde{k}_i, \tilde{x}_i)$, $\sigma_{ll} = 1$, $p_1$ such that $U_h(C^*_h) = U_h(\check{C}_1)$, and $p_2 = 0$ except for the case of $(\tilde{c}_1, \tilde{c}_2) \in \{(c_1, c_2) : 0 \leq c_1 \leq c^*_1, c_2 = f^{-1}(g(c_1))\}$ where $p_1$ and $p_2 \in (0, 1)$ such that $U_h(C^*_h) = U_h(\check{C}_1) \equiv U_h(\check{C}'_1)$. If $\lambda > \lambda^{RS}$, $c_1 > c^*_1$, and $c_2 > c^*_2$, the equilibrium is a separating equilibrium with $C_h = C^*_h$, $C_i = C^{**}_1$, $\sigma_{hl} = 0$, $\sigma_{ll} = 1$, and $p_1 = p_2 = 0$. If $\lambda < \lambda^{RS}$, $c_1 > f^{-1}(m(\lambda))$, and $c_2 \geq g^{-1}(m(\lambda))$, the equilibrium does not exist.

Despite complicated notations, the results in Proposition 4 are quite intuitive: a contract with testing can dominate other contracts only if the cost of this test is sufficiently low. Compared with the original setup in Rothschild and Stiglitz (1976) where the equilibrium exist when $\lambda \geq \lambda^{RS}$, the introduction of either underwriting or post-loss testing enlarges the range of $\lambda$ such that the equilibrium can be achieved even if $\lambda < \lambda^{RS}$. Note that when $c_1$ is larger than $c^*_1$, the profile section surface of $(c_2, \lambda)$ in Figure 6 is reduced into Figure 3 in Picard (2009). And even in the region of $c_2 > c^*_2$ and $\lambda < \lambda^{RS}$, equilibrium can be established with sufficiently low underwriting cost $c_1$. Hence, by introducing underwriting and post-loss testing simultaneously, we further extend the parameter space in Picard (2009) where the equilibrium exists in the region of high cost of post-loss test.

Once the dominating contract is supplied in the market, individuals of different types will take their optimal strategies accordingly and respectively. No other contracts can improve either type and make positive profit. Given the insureds’ strategies, insurers will not deviate to other contracts since any deviation is not profitable. In equilibrium, testing strategies are based on the insurers’ belief which is consistent with the insureds’ action. The equilibrium stated in Proposition 4, therefore, is the genuine perfect Bayesian equilibrium.

### 5 Conclusion

Underwriting and post-loss testing are commonly used screening mechanisms in insurance markets. However, due to the random and indefinite insurer-insured relationship and high turnover rate of customers in insurance companies (see Picard (2009)), along with testing costs, the insurers may not be able to pre-commit to perform both tests when individuals apply for insurance contracts. Thus, it
is of theoretical and practical importance to examine the effects of both underwriting and post-loss testing without commitment in a unified model. We build up this model in a familiar Rothschild and Stiglitz insurance market and solve it by first deriving the optimal contracts for different cases and then identifying the equilibrium by comparing the levels of utility delivered to low-risk type individuals by different contracts. In the equilibrium, whether either or both tests are used hinges on the relative magnitude of testing costs. In particular, only when the pair of cost parameters meet the rigorous conditions can both tests be used in equilibrium. In most cases, the contract for low-risk type may only incorporate the test with relatively low testing cost. Thus, the co-existence of both underwriting and post-loss testing in reality may be caused by additional reasons not analyzed in this article, such as heterogeneous risk aversion among insureds and unintentional concealment of risk types of individuals.

We show that in the semi-separating equilibrium where only underwriting is used, the insurance contract for low-risk type may be either overinsurance, full insurance, or underinsurance, dependent on the relationship between the fee charged to the misrepresenting individuals and the underwriting cost. Correspondingly, despite the mixed strategy of the high-risk type individuals, risk may be either negatively or positively correlated with coverage in our model with underwriting without commitment, which is in salient contrast to the one-side prediction in Rothschild and Stiglitz (1976) and Picard (2009). Given that testing the relationship between coverage and risk, or equivalently the existence of adverse selection, is the central topic in empirical works among insurance economists, our model provides a further factor: the difference between underwriting cost and underwriting fee, to predict this relationship. Additionally, our model suggests that when both underwriting and post-loss testing are used, the contract for low-risk type displays overinsurance while that for the high-risk type takes the form of full insurance. Potential empirical tests could be performed concerning this assertion. In summary, due to the presence of the two screening mechanisms in insurance markets, our theoretical analysis provides another answer as to why empirical research on the existence of adverse selection in insurance markets is mixed, given that underwriting and post-lose testing technology diverges among different insurance market and that the insurers may not pre-commit to perform these screening mechanisms.19

We conclude by discussing two potential extensions of our model and thus hopefully provide helpful guidance for further studies. The first recommended extension is to consider a case when contracts for both risk types being tested are updated by the insurers after they have concluded their underwriting. An additional complication is introduced in this situation because there is potential cross-subsidies between the updated contracts as long as they are provided by the same insurer. We conjecture

19See Cohen and Siegelman (2010) for an extensive survey of the adverse selection literature.
that the updated contract for the low-risk type would be the actuarially fairly priced contract of low-risk type, $C^*_l$, and thus, given contract $C^*_h$ pervades in the market, part of the individuals in the market can achieve the first-best allocation. Our second recommendation for extensions of this work is to endogenize fee $G$ collected from the misrepresenting individuals in underwriting, since in practice, $G$ can be controlled by the insurers as a part of profit management. Intuitively, a larger fee may provide more incentive to the insurers to conduct underwriting but may deter high-risk type individuals to apply for the contract designed for low-risk type individuals. The present setup in our model, however, is devoid of a mechanism to specify an optimal underwriting strategy and an optimal fee simultaneously. Hence, additional tradeoff should be introduced to solve optimal fee.
Appendix

Proof of Proposition 2

In the proof of Lemma 7, we have shown that in case of $p_1 > 0$ and $p_2 = 0$ the resource constraint of contract $C_l$ is represented by equation (24), i.e. $\psi(k_l,x_l) = 0$, which is a hyperbola on the state space. Define a transfer from $((k_l,x_l))$ to $(z,y)$ such that

$$
\begin{pmatrix}
- (1 - \pi_l) & \pi_l \\
- (1 - \pi_h) & \pi_h
\end{pmatrix}
\begin{pmatrix}
k_l \\
x_l
\end{pmatrix} =
\begin{pmatrix}
z \\
y
\end{pmatrix}.
$$

(36)

Then (24) is equivalent to:

$$
y = - \frac{c_1(c_1 - G)}{z + c_1} + c_1 - G = \frac{(c_1 - G)z}{z + c_1}.
$$

(37)

The asymptotes of the transferred hyperbola are:

$$
y = c_1 - G,
$$

(38)

and

$$
z = -c_1.
$$

(39)

Since the mapping from $((k_l,x_l))$ to $(z,y)$ is a bijection, two asymptotes of (24) are:

$$
x_l = \frac{1 - \pi_h}{\pi_h} k_l + \frac{c_1 - G}{\pi_h},
$$

(40)

and

$$
x_l = \frac{1 - \pi_l}{\pi_l} k_l - \frac{c_1}{\pi_l}.
$$

(41)

Note that $\sigma_{hl} = \frac{(1-\lambda)c_1}{\lambda(\pi_h x_l - (1-\pi_h)k_l + G - c_1)} < 1$ implies (26), which is transferred to $y > \frac{c_1}{\lambda} - G \geq c_1 - G$.

So it is sufficient to consider the upper branch of the hyperbola in $(z,y)$ space; see Figure A1. In addition, the market average resource constraint $\Xi(1,k_l,x_l) = 0$, i.e. the resource constraint when $\sigma_{hl} = 1$, is represented by:

$$
\lambda y + (1 - \lambda)z = 0,
$$

(42)

which intersects with the pair of hyperbola at points $(0,0)$ and $\left(\frac{\lambda G}{1-\lambda} - \frac{c_1}{1-\lambda}, \frac{c_1}{\lambda} - G\right)$ in $(z,y)$ space (the red dashed line in Figure A1).

Figure A1 about here.

21
Now, we focus on the interior solutions which is ensured by the condition $\lambda > \tilde{\lambda}$. Then the first order condition (25) holds.

If $c_1 > G$ and $\sigma_{kl} < 1$, the hyperbola is increasing and convex with $\frac{\lambda G}{\lambda + c_1} < z < -c_1$ and $y > \frac{\lambda}{\lambda + c_1} G \geq c_1 - G$. So we have $x_l < \frac{1}{\pi_l} k_l - \frac{c_1}{\pi_l}$ and $x_l > \frac{1}{\pi_h} k_l + \frac{c_1-G}{\pi_h}$. The slope of $\psi(k_l, x_l) = 0$ is larger than $\frac{1}{\pi_l}$. Thus, the first order condition (25) implies that $(1-\pi_l)u'(W_N-k_l) > \frac{1}{\pi_l}$, or equivalently $u'(W_N-k_l) > u'(W_A + x_l)$. since $u$ is concave, we have $W_N - k_l < W_A + x_l$. So $k_l + x_l > A$.

If $c_1 = G$, the hyperbola degenerates into part of the asymptotes with $z = -c_1$, $y > \frac{c_1}{\lambda} - G \geq c_1 - G$.

The slope of $\psi(k_l, x_l) = 0$ is exactly equal to $\frac{1}{\pi_l}$. So the first order condition (25) implies that $u'(W_N-k_l) = u'(W_A + x_l)$ and thus $k_l + x_l = A$.

If $c_1 < G$, the hyperbola is decreasing and convex. And if $c_1 > \lambda G$ we have $-c_1 < z < \frac{\lambda G}{\lambda + c_1}$ and $y > \frac{\lambda}{\lambda + c_1} G \geq c_1 - G$. If $c_1 \leq \lambda G$, we have $-c_1 < z < 0$ and $y > 0 > c_1 - G$ for $k_l > 0$ and $x_l > 0$. In both cases, we have $x_l > \frac{1}{\pi_l} k_l - \frac{c_1}{\pi_l}$ and $x_l > \frac{1}{\pi_h} k_l + \frac{c_1-G}{\pi_h}$. The slope of $\psi(k_l, x_l) = 0$ is lower than $\frac{1}{\pi_l}$. So the first order condition (25) implies that $(1-\pi_l)u'(W_N-k_l) < \frac{1}{\pi_l}$, or equivalently $u'(W_N-k_l) < u'(W_A + x_l)$. since $u$ is concave, we have $W_N - k_l > W_A + x_l$. So $k_l + x_l < A$.

Finally, since we always have $z < 0$ if $c_1 > 0$, the resource constraint with underwriting will always below the fair-odds line of l-type, irrelevant to the magnitude of $G$.

**Derivation of Equation (31)**

By transfer (36), we have shown in the proof of Proposition 2 that $\Xi(\sigma^u(k_l, x_l), k_l, x_l) = 0$ is equivalent to (37). With the same transfer, $\Xi(\sigma^f(k_l, x_l), k_l, x_l) = 0$ is equivalent to:

$$y = \frac{\pi_h(1-\pi_h)z^2 + (\pi_h - \pi_l)\pi_h c_2 z}{\pi_h(1-\pi_l)z + (\pi_h - \pi_l)\pi_1 c_2}. \quad (43)$$

Note that untrivial solutions to $(k_l, x_l)$ are also untrivial solutions to $(z, y)$. So, combining (37) and (43) and cancelling out $z$, we have:

$$0 = \pi_h(1-\pi_h)z^2 + ((\pi_h - \pi_l)\pi_h c_2 + \pi_h(1-\pi_h)c_1 - \pi_h(1-\pi_l)(c_1-G))z$$
$$+ (\pi_h - \pi_l)^2 c_1 c_2 + (\pi_h - \pi_l)\pi_1 G c_2, \quad (44)$$

where a factor of $z$ has been cancelled. If (30) has one and only untrivial solution, then the determinant of (44) must be 0, which is equation (31). Conversely, if equation (31) holds, there is unique pair $(k_l, x_l) \neq 0$ such that two hyperbolas in (30) has one and only tangent point.
Proof of Proposition 3

By reasoning before Proposition 3, we have shown that problem (14) with \( p_1 > 0 \) and \( p_2 > 0 \) is equivalent to a system of equations (28), (29), and (31). Equation (28) is equivalent to

\[
\frac{c_1}{\pi_h x_l - (1 - \pi_h)k_l + G - c_1} = \frac{\pi_t c_2}{\pi_h (x_l - c_2)}, \tag{45}
\]

and equation (29) implies that:

\[
\frac{1 - \pi_h) c_1 - \pi_h c_1 s}{(\pi_h x_l - (1 - \pi_h)k_l + G - c_1)} = -\frac{\pi_t c_2 s}{\pi_h (x_l - c_2)^2}. \tag{46}
\]

Square equation (45) and divide it by equation (46), and we have that (35). And (35) together with (34) determines a pair of cost parameters \((c_1, c_2) = (c_1', c_2')\).

It is easy to show that \( \varphi(c_1, c_2) = 0 \) is a hyperbola if \( G > 0 \).\(^{20}\) With little loss of generality, assume that \( G > 0 \). Clearly, we have \( c_2 \geq 0 \) if \( c_1 \geq 0 \) by (31). When \( c_1 = G \), we have \( c_2 = \frac{1 - \pi_h}{\pi_h - \pi_t} G \). By calculation, \( \frac{\partial \varphi(G, (1 - \pi_h) G / (\pi_h - \pi_t))}{\partial c_2} = 0 \). So, \( \frac{dc_2}{dc_1} = +\infty \) at \( G, \frac{1 - \pi_h}{\pi_h - \pi_t} G \) on \( \varphi(c_1, c_2) = 0 \). Besides, two asymptotes\(^{21}\) of the hyperbola \( \varphi(c_1, c_2) = 0 \) are:

\[
c_2 = \frac{\pi_h}{(1 - \sqrt{1 - \pi_h})^2} c_1 + \frac{\pi_h \sqrt{1 - \pi_h} - \pi_h (1 - \pi_l)}{(\pi_h - \pi_t) (1 - \sqrt{1 - \pi_h})} G, \tag{47}
\]

and

\[
c_2 = \frac{\pi_h}{(1 + \sqrt{1 - \pi_h})^2} c_1 - \frac{\pi_h \sqrt{1 - \pi_h} + \pi_h (1 - \pi_l)}{(\pi_h - \pi_t) (1 + \sqrt{1 - \pi_h})} G. \tag{48}
\]

And for \( c_1 \geq G \), the branch of hyperbola \( \varphi(c_1, c_2) = 0 \) lies below (47) and above (48). If \( \tilde{C}_1' \) is the solution to problem (14), \( z \) must be smaller than \( c_1 \) as shown in the proof of Proposition 2, i.e.

\[
-c_1 - z = \frac{(\pi_h - \pi_t) c_2 - (1 - \pi_h) c_1 - (1 - \pi_l) (c_1 - G)}{2(1 - \pi_h)} > 0. \tag{49}
\]

So we have \( (\pi_h - \pi_t) c_2 - (1 - \pi_h) c_1 > (1 - \pi_l) (c_1 - G) \geq 0 \). Then, \( \frac{c_2}{c_1} > \frac{1 - \pi_h}{\pi_h - \pi_t} \). Thus, \((c_1', c_2')\) must be above \( G, \frac{1 - \pi_h}{\pi_h - \pi_t} G \) on \( \varphi(c_1, c_2) = 0 \); see Figure A2.

Figure A2 about here

Conversely, we are going to show firstly that if

\[
\sqrt{1 - \pi_h} > 1 - \pi_t, \tag{50}
\]

\(^{20}\)When \( G = 0 \), the hyperbola \( \varphi(c_1, c_2) = 0 \) is reduced into its asymptotes presented below.

\(^{21}\)This can be calculated by a linear transfer:

\[
\begin{pmatrix}
-\pi_h (\pi_h - \pi_t) \\
0
\end{pmatrix}
\begin{pmatrix}
2 - \pi_h (\pi_h - \pi_t) \\
\pi_h - \pi_t
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
+ \begin{pmatrix}
1 - \pi_h \\
\pi_h (1 - \pi_t)
\end{pmatrix} G
= \begin{pmatrix}
q \\
r
\end{pmatrix}.
\]

Then two asymptotes of the transferred hyperbola are \( q = \pm 2 \sqrt{1 - \pi_h} r \).
If $G > l$-type that goes through $\tilde{\sigma}$

Rearranging (35), we have

$$\sigma c \text{ hold, then } (\text{the equation for the tangent line must be tangent to that of (43) at some point such that } z < \text{Condition 2 are satisfied. If } \sigma \sigma \sigma (0 \pi c \pi l 1', x) \text{ is above (48). Next, (51) and (52) show that:}

$$\frac{1 - \pi(h)\pi(s - (1 - \pi(h))\pi(l)}{\pi(h)\pi(s - (1 - \pi(h))\pi(l) > 1 - \pi(h)}$$

and

$$\frac{\pi(h)s}{\pi(h)\pi(s - (1 - \pi(h))\pi(l)} > 1 - \pi(h)$$

hold, then $(c_1, c_2) = (\tilde{c}_1', \tilde{c}_2')$ solved by (34) and (35) exist and then show that Condition 1 and Condition 2 are satisfied. If $G = 0$, $(\tilde{c}_1', \tilde{c}_2') = (0, 0)$ solves system (34) and (35); this case is trivial.

If $G > 0$, (50) implies that $\frac{1}{2}\pi(h) - \pi(h)(1 - \pi(h)) > 0$. Since we always have $\frac{(\pi(h) - \pi(l))(1 - \pi(h))^{1/2}}{2\pi(h)} > 0$, then

$$\frac{\pi(h)\pi(s - (1 - \pi(h))\pi(l)}{(\pi(h) - \pi(l))(1 - \pi(h))} > 0. \text{ So (0, 0) is below (47). On the other hand, } -\frac{\pi(h)\pi(s - (1 - \pi(h))\pi(l)}{(\pi(h) - \pi(l))(1 + \sqrt{1 - \pi(h)})} G < 0, \text{ so (0, 0) is above (48). Next, (51) and (52) show that:}

$$\frac{1 - \pi(h)}{\pi(h) - \pi(l)} < \frac{\pi(h)s}{\pi(h)\pi(s - (1 - \pi(h))\pi(l)} < \frac{\pi(h)}{(1 - \sqrt{1 - \pi(h)})^2}.$$  

Thus, line $c_2 = \frac{\pi(h)\pi(s - (1 - \pi(h))\pi(l)}{\pi(h)\pi(s - (1 - \pi(h))\pi(l) c_1}$ must intersect with hyperbola $\varphi(c_1, c_2) = 0$ at some point denoted by $(\tilde{c}_1', \tilde{c}_2')$ above $G, \frac{1 - \pi(h)\pi(s - (1 - \pi(h))\pi(l)}{\pi(h) - \pi(l)} G$. At $(\tilde{c}_1', \tilde{c}_2')$, the determinant of (44) is equal to 0. Thus, the curve of (37) must be tangent to that of (43) at some point such that $z < 0$. Equivalently, given $(c_1, c_2) = (\tilde{c}_1', \tilde{c}_2')$, (30) has one and only untrivial solution $(k_1, x_1)$ at which the upper branch of $\Xi(\sigma^U(k_1, x_1), k_1, x_1) = 0$ is tangent to the upper branch of $\Xi(\sigma^P(k_1, x_1), k_1, x_1) = 0$. So Condition 1 is satisfied. In particular, the equation for the tangent line is $\sigma^U(k_1, x_1) - \sigma^P(k_1, x_1) = 0$, i.e.:

$$x_1 = \frac{(1 - \pi(h))\pi(l)\tilde{c}_2 - \pi(h)c_1}{\pi(h)\pi(l)\tilde{c}_2 - \pi(h)c_1} k_1 - \frac{\pi(h)\pi(l)\tilde{c}_2 G + (\pi(h) - \pi(l))\tilde{c}_1'\tilde{c}_2'}{\pi(h)\pi(l)\tilde{c}_2 - \pi(h)c_1}. $$

Rearranging (35), we have

$$s = \frac{(1 - \pi(h))\pi(l)\tilde{c}_2}{\pi(h)\pi(l)\tilde{c}_2 - \pi(h)c_1}. $$

Since the slope of $\Xi(\sigma^P(k_1, x_1), k_1, x_1) = 0$ is monotonic, the tangent line of $\Xi(\sigma^P(k_1, x_1), k_1, x_1) = 0$ and $\Xi(\sigma^U(k_1, x_1), k_1, x_1) = 0$ coincide with that of $\Xi(\sigma^U(k_1, x_1), k_1, x_1) = 0$ and an indifference curve of l-type that goes through $\tilde{C}_1'$. Then, Condition 2 is satisfied.

---

22If $(k_1, x_1)$ and $(k_1', x_1')$ are on the secant line of $\Xi(\sigma^U(k_1, x_1), k_1, x_1) = 0$ and $\Xi(\sigma^P(k_1, x_1), k_1, x_1) = 0$, we must have $\sigma^U(k_1, x_1') = \sigma^P(k_1, x_1')$ and $\sigma^U(k_1', x_1') = \sigma^P(k_1', x_1')$. Thus, the equation for the secant line of both hyperbolas is $\sigma^U(k_1, x_1') - \sigma^P(k_1, x_1') = 0$. When $\Xi(\sigma^U(k_1, x_1), k_1, x_1) = 0$ is tangent to $\Xi(\sigma^P(k_1, x_1), k_1, x_1) = 0$, the secant line is reduced into the tangent line.
References


This figure plots basic contracts in the model in the state space of \((W_1, W_2)\). The dash-dot line is the 45° line. Point E stands for the state where no insurance is purchased. Line EH is the fair-odds line for the h-type individuals, and line EL is that for the l-type individuals. Contract \(C^*_h\) is located at the tangent point of indifference curve of h-type and their fair-odds line. That indifference curve intersects with the fair-odds line of l-type at contract \(C^*_{l*}\). \(C^*_h\) and \(C^*_{l*}\) are separating contracts for h-type and l-type respectively in Rothschild and Stiglitz (1976). Contract \(C^*_l\) is located at the tangent point of indifference curve of l-type and their fair-odds line. In this figure, \(W_N = 100, \ W_A = 20, \ \pi_h = 0.7, \ \pi_l = 0.4, \text{ and } u(w) = \ln(w).\) So \(C^*_h = (56, 24), \ C^*_l = (32, 48), \text{ and } C^*_{l*} = (8.0548, 12.0821).\)
This figure plots optimal contract with underwriting, \( \tilde{C}_l \), in the model in the state space of \((W_1, W_2)\). In each panel, the dash-dot line is the \(45^\circ\) line. Point E stands for the state where no insurance is purchased. Line EH is the fair-odds line for the h-type individuals, and line EL is that for the l-type individuals. The dark solid curve is the resource constraint \( \psi(W_N - W_1, W_2 - W_A) = 0 \). And the optimal contract \( \tilde{C}_l \) is located at the tangent point of indifference curve of l-type and the budget constraint. In this figure, \( W_N = 100, W_A = 20, \pi_h = 0.7, \pi_l = 0.4, \) and \( u(w) = \ln(w) \).

In Panel A, \( c_1 = 3 > 2 = G \), the optimal contract \( \tilde{C}_l = (35.3944, 45.2172) \) (overinsurance), and \( \sigma_{hl} = 0.0166, p_1 = 0.5034 \). In Panel B, \( c_1 = 2 = G \), the optimal contract \( \tilde{C}_l = (34, 46) \) (full insurance), and \( \sigma_{hl} = 0.0101, p_1 = 0.5127 \). And in Panel C, \( c_1 = 1 < 2 = G \), the optimal contract \( \tilde{C}_l = (32.9003, 46.9546) \) (underinsurance), and \( \sigma_{hl} = 0.0046, p_1 = 0.5217 \).
In this figure, contract $\tilde{C}_l$ is the optimal contract with underwriting derived in Proposition 2, which is the tangent point of the resource constraint $\psi(W_N - W_1, W_2 - W_A) = 0$ and indifference curve of l-type. If the resource constraint with post-loss test intersects with the resource constraint with underwriting at $\tilde{C}_l$, there exists a contract with only post-loss test, denoted by $C_l$, which is located below the resource constraint with post-loss test but above the resource constraint with underwriting, that if provided, can improve the utility of at least l-type and makes positive profit.
This figure plots optimal contracts with both underwriting and post-loss test, $\tilde{C}'_l$, in the model in the state space of $(W_1, W_2)$. The dash-dot line is the $45^\circ$ line. Point E stands for the state where no insurance is purchased. Line EH is the fair-odds line for the h-type individuals, and line EL is that for the l-type individuals. The dark solid curve is the resource constraint with underwriting $\Xi(\sigma^U(k_l, x_l), k_l, x_l) = 0$. And the cyan solid curve is the resource constraint with post-loss test $\Xi(\sigma^P(k_l, x_l), k_l, x_l) = 0$. The optimal contract $\tilde{C}'_l$ is located at the tangent point between indifference curve of l-type and both resource constraints. To this end, the parameters are selected so that sufficient conditions in Proposition 3 are satisfied. In particular, in this figure, $W_N = 100, W_A = 20, \pi_h = 0.7, \pi_l = 0.5, G = 2$ and $u(w) = \ln(w)$. Cost parameters are solved as $(\tilde{c}_1, \tilde{c}_2) = (3.9772, 13.2651)$, and contract for l-type is $\tilde{C}'_l = (46.6434, 37.1181)$. Also, $s = 1.0705, C'_h = (56, 24), C'_l^{**} = (12.8239, 12.8239)$, and $\sigma_{hl} = 0.0441$. 
This figure illustrates the relationship between $\lambda^{RS}$ and $\tilde{\lambda}$ when the contract with underwriting, $\tilde{C}_l$, dominates the separating contract $C_l^{**}$ in the state space of $(W_1, W_2)$. The dash-dot line is the $45^\circ$ line. Point E stands for the state where no insurance is purchased. Line EH is the fair-odds line for the h-type individuals, and line EL is that for the l-type individuals. The dark solid curve is the resource constraint with underwriting $\psi(W_N - W_1, W_2 - W_A) = 0$. The market fair-odds line with $\lambda = \tilde{\lambda}$ (the cyan dotted line) intersects point E and contract $\tilde{C}_l$. And the market fair-odds line with $\lambda = \lambda^{RS}$ (the gray dotted line) intersects point E and is tangent to an indifference curve that intersects contract $C_l^{**}$. For $\tilde{C}_l$ to be a interior solution, the market average fair-odds line must be lower than that with $\lambda = \tilde{\lambda}$. To exclude a pooling contract that may upset the separating equilibrium, the market average fair-odds line must be lower than than with $\lambda = \lambda^{RS}$. Since the indifference curve of l-type that intersects $C_l^{**}$ is lower than that which intersects $\tilde{C}_l$, we have $\lambda^{RS} > \tilde{\lambda}$. In this figure, $W_N = 100$, $W_A = 20$, $\pi_h = 0.7$, $\pi_l = 0.5$, $G = 2$, $c_1 = 3$ and $u(w) = \ln(w)$. We have $\lambda^{RS} = 0.4651$ and $\tilde{\lambda} = 0.2008$. 

FIGURE 5
Illustration of $\lambda^{RS}$ and $\tilde{\lambda}$
This figure identifies which contract dominates in the market by comparing utility of l-type that can be provided by different contracts. In particular, $g(c_1) \equiv U_l(\tilde{C}_l)$, $f(c_2) \equiv U_l(\hat{C}_l)$, and $m(\lambda)$ denotes the utility of l-type derived from a pooling contract without tests that maximizes their utility. In both panels, the red solid curve stands for the function $f(c_2) = g(c_1)$. We set $W_N = 100$, $W_A = 20$, $\pi_b = 0.7$, $\pi_l = 0.5$, $G = 2$, and $u(w) = \ln(w)$. Then we have $\lambda^{RS} = 0.4651$, $(c_1^*, c_2^*) = (4.7857, 16.6421)$, $(\tilde{c}_1^*, \tilde{c}_2^*) = (3.9772, 13.2651)$. Panel A presents the case with $\lambda \geq \lambda^{RS}$ and Panel B presents the case with $0.2167 = \lambda < \lambda^{RS}$. 
This figure presents the transferred resource constraint (37) in $(z, y)$ space in different cases with respective to relative magnitude of $c_1$ and $G$. The transferred resource constraint should be above the market average fair-odds line so that problem (14) with $p_1 > 0$ can be solved by an interior solution, and a pooling contract with $\sigma_{hl} = 1$ is excluded. In this figure, $\lambda = 0.85$, $G = 2$, and $c_1 = 3$, 2, and 1 in three panels respectively.
This figure plots parabola $\varphi(c_1, c_2) = 0$ with parameters $\pi_h = 0.7$, $\pi_l = 0.4$, and $G = 2$. If pair $(\tilde{c}_1', \tilde{c}_2')$ are located above $\left( G, \frac{1-\pi_h}{\pi_h-\pi_l} G \right)$ on $\varphi(c_1, c_2) = 0$, then $z < -c_1$ and so the solution to the contract is economically meaningful. Otherwise, $z \geq -c_1$. It is worth noting that in this figure, $(0,0)$ is above (47). Thus, condition (50) fails. Even this, $(\tilde{c}_1', \tilde{c}_2')$ solving (34) and (35) may still exist such that contract $\tilde{C}_l'$ is the solution to problem (14) with both tests. Thus, conditions (50), (51), and (52) are only sufficient conditions instead of necessary conditions.