A Cautionary Note on Pricing Longevity Index Swaps
(Joint work with Johnny S.H. Li)

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Objectives

- Pricing QxX index swap
- Examining the parameter risk and model risk in the pricing
- Determining the effect of the uncertainty on the pricing

Outline

- Mortality derivatives
- QxX index Swap
- Parameter risk
- Model risk
- Conclusion
Mortality Derivatives

What are mortality derivatives?

- Financial contracts that have payoffs tied to the level of a certain longevity or mortality index
- Examples: survivor bond, survivor swap, ...

How to price mortality derivatives?

- Mortality model
- Wang’s Transform, Q measure, ...
A two-factor stochastic mortality model (Cairns, Blake and Dowd (2006))

Mathematical Specification:

\[
\ln \frac{q_{x,t}}{1 - q_{x,t}} = A_1(t) + A_2(t)x. \tag{1}
\]

- \(x\) → age
- \(t\) → time
- \(q_{x,t}\) → realized single-year death probability
- \(\{A_1(t)\}\) and \(\{A_2(t)\}\) → discrete-time stochastic processes
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Mortality Derivatives

Mortality model

A two-factor stochastic mortality model (con’t)

Stochastic Mortality: Recall: \( \ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \)

\[
D(t + 1) = A(t + 1) - A(t) \\
= \mu + CZ(t + 1)
\]  

\( A(t) = (A_1(t), A_2(t))' \)

\( \mu \rightarrow \) constant 2 \( \times \) 1 vector

\( C \rightarrow \) constant 2 \( \times \) 2 upper triangular matrix

\( Z(t) \rightarrow \) 2-dim standard normal random variable
Model fitting

Data

- $q_{x,t}$, $x = 65, 66, \ldots, 109$, $t = 1971, 1972, \ldots, 2005$

Model fitting

$$\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \quad D(t + 1) = \mu + CZ(t + 1)$$

- First step: Estimate $A(t)$ by least square method
- Second step: Estimate $\mu$ and $C$ through maximum likelihood estimation
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Mortality Derivatives

Mortality model

Forecasting

Steps

\[
\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \quad D(t+1) = \mu + CZ(t+1)
\]

► Simulate a set of \( Z \)
► Obtain corresponding \( D(2005 + k), \quad k = 1, 2, \ldots, 10 \)
► \( A(2005 + k) = A(2005) + \sum_{n=1}^{k} D(2005 + n), \quad k = 1, 2, \ldots, 10 \)
► Calculate \( q_{x,2005+k} \)
Pricing in Risk-adjusted world

Real-world probability measure (P measure)

\[ D(t + 1) = \mu + CZ(t + 1) \]  \hspace{1cm} (3)

Risk-adjusted probability measure (Q measure)

\[
\begin{align*}
D(t + 1) &= \mu + C(\tilde{Z}(t + 1) - \lambda) \\
       &= \tilde{\mu} + C\tilde{Z}(t + 1),
\end{align*}
\]  \hspace{1cm} (4)

where \( \lambda \) is the market price of risk and \( \tilde{\mu} = \mu - C\lambda \).
QxX Index

“allows market participants to measure, manage and trade exposure to longevity and mortality risks in a standardized, transparent, and real-time manner"

- Launched by Goldman Sachs in 2007
- Based on a reference pool consisting of a set of lives underwritten by AVS Underwriting LLC
- The index value is the number of lives in the reference pool
- Published monthly, providing “real-time" mortality information
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Payment structure of QxX index swap

\[ X \left( \frac{S_{k-1}}{S_0} \cdot \frac{\sigma}{12} \right) \]

\[ X \left( \frac{S_{k-1} - S_k}{S_0} \right) \]

- \( X \rightarrow \) nominal amount
- \( S_k \rightarrow \) index value in the \( k \)th month
- \( \sigma \rightarrow \) fixed spread
- Goldman Sacs: \( \sigma = 500 \) basis points for 10-year swap
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QxX index swap

Pricing a 10-year QxX index swap

10-year QxX index swap price

- QxX index swap is priced by determining the “fair” spread $\sigma$
  
  $\text{Market value of future payments from fixed payer} = \text{Market value of future payments from fixed receiver}$

- We need to know the market price of risk $\lambda$. In our analysis,
  - Not enough data to estimate $\lambda$ for QxX index swaps
  - Use the estimated market price of risk from BNP/EIB longevity bond
10-year QxX index swap price (Con’t)

Estimates of $\sigma$ (in basis points) under different choices of

$\lambda = (\lambda_1, \lambda_2)$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>0</td>
<td>627</td>
</tr>
<tr>
<td>0</td>
<td>0.316</td>
<td>619</td>
</tr>
<tr>
<td>0.175</td>
<td>0.175</td>
<td>622</td>
</tr>
</tbody>
</table>

Why $\sigma \neq 500$ bps?

- No access to the actual QxX index reference pool
- Lack of market data for the swap
- Existence of parameter risk and model risk
Parameter risk under Bayesian Method

- $D(t) \sim \text{MVN}(\mu, V)$, where $V = C' C$.
- Treat $\mu$ and $C$ as random variables
  \[ D(t) \mid \mu, V \sim \text{MVN}(\mu, V) \]  \hspace{1cm} (5)
- Use a non-informative prior distribution
  \[ \pi(\mu, V) \propto |V|^{-3/2} \]  \hspace{1cm} (6)
- Marginal posterior distribution
  \[ V^{-1} \mid D \sim \text{Wishart}(n - 1, n^{-1} \hat{V}^{-1}) \]  \hspace{1cm} (7)
  \[ \mu \mid D \sim \text{MVN}(\hat{\mu}, n^{-1} \hat{V}) \]
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Parameter risk

Bayesian Method

Estimated marginal posterior density functions for the model parameters

Figure: Simulated marginal posterior parameter distributions. (We denote the $i$th element in $\mu$ by $\mu_i$ and the $(j, k)$th element in $V$ by $V_{j,k}$).
Simulated predictive distribution of $\sigma$, $\lambda = (0.375, 0)$
95% Confidence Interval for $\sigma$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>With parameter risk</th>
<th>Without parameter risk</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>0</td>
<td>(560,693)</td>
<td>(574,680)</td>
</tr>
<tr>
<td>0</td>
<td>0.316</td>
<td>(553,685)</td>
<td>(567,673)</td>
</tr>
<tr>
<td>0.175</td>
<td>0.175</td>
<td>(557,686)</td>
<td>(571,675)</td>
</tr>
</tbody>
</table>

**Table:** 95% confidence intervals for $\sigma$ (in basis points) under different choices of $\lambda_1$ and $\lambda_2$. 
Model risk in pricing

Figure: Estimated values of $A_1(t)$ and $A_2(t)$, 1971–2005.
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Model risk

Reason for the reverse trend

What causes the reverse trend?

Crude mortality curves

\[ \ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x \]
What causes the reverse trend?

Life expectancies at age 65

\[
\ln \frac{q_{x,t}}{1-q_{x,t}} = A_1(t) + A_2(t)x
\]
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Future trends

Three possible scenarios

![Graph showing three possible scenarios with data points for A1(t) and A2(t) over the years 1960 to 2020.](image)
### How does the change affect QxX index swap price?

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>Scenario 1</th>
<th>Scenario 2</th>
<th>Scenario 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>0</td>
<td>627</td>
<td>674</td>
<td>566</td>
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<td>0</td>
<td>0.316</td>
<td>619</td>
<td>683</td>
<td>553</td>
</tr>
<tr>
<td>0.175</td>
<td>0.175</td>
<td>622</td>
<td>678</td>
<td>558</td>
</tr>
</tbody>
</table>

**Table**: Swap spread (in basis points) under three different scenarios.
Conclusion

- The swap spread computed from our pricing framework is fairly close to the spread currently offered by Goldman Sachs.
- The pricing is still very experimental.
  - Parameter risk and model risk are significant in the pricing.
  - No sufficient market price data to estimate market prices of risk.
  - No clear conclusion on how mortality rates may evolve in the future.