Regime-Switching Portfolio Replication

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1 Investment Guarantees
   - Portfolio Replication
   - The S&P 500

2 Regime Switching Portfolio Replication
   - Hidden Markov Models
   - Regime Switching Replication

3 Regime Switching Bayesian Portfolio Replication
   - Bayesian Estimation of Regime Switching Models
   - Example Results
   - Conclusion and Future Work
Long-Term Guarantees

**Contract:** Long-term Equity Guarantees/Options

Eg.
- Guaranteed Minimum Maturity Benefit
- Long-Term Stock Options


Due to the catastrophe nature of this risk, choose to hedge the contract.
Black-Scholes Hedging

Black-Scholes Put Option Price:

\[
BSP_t = K \cdot e^{-r(T-t)} \cdot \Phi(-d_2) - S_t \cdot \Phi(-d_1)
\]

\[
d_1 = \frac{\log(S_t/K) + (T-t)(r + \sigma^2/2)}{\sqrt{T-t}\sigma}
\]

\[
d_2 = d_1 - \sigma \sqrt{T-t}
\]

Hedge: Hold \( H_t = -\Phi(-d_1) \) in stock.

One assumption of the framework: continuous re-balancing of the hedge.
Continuous re-balancing is obviously not feasible.

Monthly Re-balancing

- This will introduce Hedging Error
- \( HE_{t+1} = BSP_{t+1} - (H_t \cdot S_{t+1} + B_t \cdot e^r) \)

Another assumption: \( S_t \) follows a geometric Brownian Motion with constant variance \( \sigma^2 \).

Goal: Find a good \( \sigma \) for the S&P 500.
Figure: S&P 500 Monthly Index and Log-Return Levels
One could just estimate the volatility of the entire process.

- Such an approach would not capture the volatility clustering of the process.

A better approach would be to let the volatility parameter change over time, mimicking the volatility of the index.

Approach: Use a model that captures the volatility clustering of the index.
Suppose we have a time series that from \( t = 1, 2, \ldots, t_0 \) is governed by

\[
y_t = \mu_1 + \sigma_1 \epsilon_t
\]

At time \( t_0 \), there was a significant change in the parameters of the series. Over \( t_0, \ldots, t_1 \), the series behaves as

\[
y_t = \mu_2 + \sigma_2 \epsilon_t
\]

Then, at \( t_1 \), it changes back.
Hamilton (1989) proposed hidden Markov models for financial applications.

The idea being the market passes through different states:
- A stable normal market
- A high-volatility market
- Periods of uncertainty in transition between the above two states

Hidden Markov models can capture volatility clustering through the underlying state process.
Regime Switching Model Characteristics:

- The distribution of $Y_t$ is only known conditional on $\rho_t \in \{1, 2, \ldots, K\}$, the regime of the process at time $t$.
- The unobserved regime process is Markov.
- The one-period transition probabilities are defined as

$$p_{i,j} = P[\rho_t = j|\rho_{t-1} = i] \quad \forall i, j \in \{1, 2, \ldots, K\}, \forall t \in \{1, 2, \ldots, T\}$$

RSLN-2 Model: $Y_t = \log(S_t/S_{t-1})$

$$Y_t|\rho_t = \mu_{\rho_t} + \sigma_{\rho_t} \cdot \epsilon_t$$

$$\rho_t|\rho_{t-1} = k \quad \text{w.p.} \quad p_{\rho_{t-1},k} \quad k \in \{1, 2\}$$
RSLN-2 Model for the S&P 500

Maximum Likelihood Parameters for the S&P 500:

<table>
<thead>
<tr>
<th>Regime</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>Transition Parameters</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>0.00990</td>
<td>0.03412</td>
<td>$p_{1,2} = 0.0475$</td>
<td>$\pi_1 = 0.809$</td>
</tr>
<tr>
<td>Two</td>
<td>-0.01286</td>
<td>0.06353</td>
<td>$p_{2,1} = 0.2017$</td>
<td>$\pi_2 = 0.191$</td>
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Generating a Volatility from the RSLN-2 Model

Static Unconditional Volatility

\[ \sigma = \sqrt{\text{Var}[Y_t]} = \sqrt{\text{Var}[E[Y_t|\rho_t]] + E[\text{Var}[Y_t|\rho_t]]} \]

using the \( \pi_k \)'s as regime weights.

This approach seems counterproductive:

- If one went to all the trouble of modeling volatility clustering, why use a static volatility?

Need to use the information in the data to more accurately select a volatility.
The recent data observations provide insight into the current regime of the process.

Data-dependent Regime Probabilities:

\[ p_k(t) = Pr(\rho_t = k | y_t, \ldots, y_1) \]
Future Data-dependent Regime Probabilities

\[ p_k^+(t) = Pr(\rho_{t+1} = k|y_t, \ldots, y_1) = p_1(t) \cdot p_{1,k} + p_2(t) \cdot p_{2,k} \]

Question: How best can these probabilities be used in portfolio replication?
Generating a Volatility from the RSLN-2 Model

Dynamic Unconditional Volatility

\[ \sigma = \sqrt{\text{Var}[Y_t]} \]
\[ = \sqrt{\text{Var}[E[Y_t|\rho_t]] + E[\text{Var}[Y_t|\rho_t]]} \]

using the \( p_k^+(t) \)'s as regime weights.

- If the model is ‘correct’, this is the unconditional volatility of the upcoming observation.
- The regime will be one or the other; the dynamic volatility will generally not be equal to either of the regime volatilities.
- But, you’re somewhat covered against the less likely regime.
Indicator Volatility

\[ \sigma = \sigma_k, \quad \text{where } p_k^+(t) = \max(p_1^+(t), p_2^+(t)) \]

- If the model is ‘correct’, this method will pick the correct volatility often.
- But, when you’ve picked the wrong regime, your volatility is significantly off.
One observation about the two methods:

- The change in hedging volatility significantly affects your monthly hedging error.

- The Dynamic Volatility method has the largest number of significant jumps.

- The Indicator method has the biggest jumps, but less of them.

Question: Which of these hedging options is better?
Regime Switching Optimization Methods

Answer: It’s actually option dependent.

S&P 500 10-Year Put Example

- Strike Price = $S_0 = 100$
- Monthly re-balancing.
- Bond: 5% per annum.
- Transaction Costs: 0.02% of change in stock position
- Using the described hedging methods, simulate from the model to determine which method generates the smaller total option costs (initial hedge + hedging error)
The Indicator method performs exceptionally well (19%!). But why?
The Dynamic and Indicator methods perform very similarly in most cases.

When moving from Regime 2 to Regime 1, the Dynamic is too slow to react.
What About Parameter Uncertainty?

Parameter uncertainty is an important consideration

- Quite important for the example since I simulated from the fitted model to obtain results.
- Especially for Regime-switching models

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Bayesian Modeling

- Treat each parameter as itself a random variable.
- Model beliefs about each parameter using prior distributions.
- Update your distributions based on the data to form posteriors.

For the RSLN-2, Metropolis-Hastings Algorithm was used

- Very quick simulation
RSLN-2 Parameter Comparison

Maximum Likelihood Parameters for the S&P 500:

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<tr>
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<td>0.0341</td>
<td>$p_{1,2} = 0.0475$</td>
</tr>
<tr>
<td>Two</td>
<td>-0.0129</td>
<td>0.0635</td>
<td>$p_{2,1} = 0.2017$</td>
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Bayesian Posterior-Means for the S&P 500:

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</tr>
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<tbody>
<tr>
<td>One</td>
<td>0.0099</td>
<td>0.0340</td>
<td>$p_{1,2} = 0.0620$</td>
</tr>
<tr>
<td>Two</td>
<td>-0.0129</td>
<td>0.0652</td>
<td>$p_{2,1} = 0.2631$</td>
</tr>
</tbody>
</table>
RSLN-2 Parameter Posterior Distributions

- P12
- P21
- Mu1
- Mu2
- Sigma1
- Sigma2
10-Year S&P Put Example

S&P 500 10-Year Put Example Revisited

- Use the posterior parameter distributions to generate the model simulations
- Still use the MLE parameter estimates for hedging decisions.
10-Year S&P Put Example Revisited

The Indicator still performs best, but by less of a margin (16%).

<table>
<thead>
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<th>Volatility</th>
<th>Static</th>
<th>Dynamic</th>
<th>Indicator</th>
</tr>
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<tbody>
<tr>
<td>EPV[Total Option Cost]</td>
<td>3.0107</td>
<td>2.8290</td>
<td>2.5306</td>
</tr>
</tbody>
</table>
Conclusions & Future Work

Summary of Results:

- Regime-switching portfolio replication can be worth it.
- Best type of method depends on the option you’re hedging.
- Often, you want hedging strategies that react quickly.
- Parameter uncertainty can play a role.

Future Work

- More complicated Regime-Switching or Hybrid Models (RSGARCH)
- Relax the fixed interest rate assumption