The Distribution of The Total Dividend Payments in a MAP Risk Model with Multi-Threshold Dividend Strategy

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44th ARC, Madison, 2009

This is the joint work with Dr. Yi Lu, SFU
Outline of Topics

1. Introduction
2. Differential Approach
3. Layer-Based Recursive Approach
4. Numerical Example
5. Conclusion
Sample Surplus Process

Surplus $U(t)$

Premium rate $= c$

Time $t$

premiums

claims

ruin

$u$

$0$
The Classical Risk Model

- The surplus process \( \{U(t); t \geq 0\} \) with \( U(0) = u \), s.t.
  \[
dU(t) = cdt - dS(t), \quad t \geq 0.
\]

- Premiums are collected continuously at a constant rate \( c \)

- A sequence of non-negative claim amounts r.v. \( \{X_n; n \in \mathbb{N}^+\} \)

- Number of claims up to time \( t \), \( N(t) \sim \text{Poisson}(\lambda t) \)

- Aggregate claim amounts up to time \( t \), \( S(t) = \sum_{n=1}^{N(t)} X_n \)

- Time of ruin \( \tau = \inf\{t \geq 0 : U(t) < 0\} \)
MAP Risk Model

MAP \((\vec{\alpha}, D_0, D_1)\)

- Initial distribution, \(\vec{\alpha}\)
- Intensity matrix, \(D_0 + D_1\)
- Intensity of state changing without claim, \(D_0(i, j) \geq 0, j \neq i\)
- Intensity of state changing with claim, \(D_1(i, j) \geq 0\)
- The diagonal elements of \(D_0\) are negative values, s.t. \(D_0 + D_1 = 0\)

- Special cases: classical risk model, Sparre-Andersen risk model, Markov-modulated risk model

Reference: Badescu et al. (2007), Badescu (2008), Ren (2009),
Various Dividend Strategies

![Graph showing surplus over time with premium and claim lines, indicating ruin at a certain point.]

- Various Dividend Strategies
- Time t
- Premiums
- Claims
- Ruin
- Surplus
- Time t
Various Dividend Strategies
Various Dividend Strategies

- Various Dividend Strategies
- Differential Approach
- Layer-Based Recursive Approach
- Numerical Example
- Conclusion

The diagram illustrates the surplus process over time $t$, with premium payments and claims. The surplus levels $b_1$ and $b_2$ are shown, with the point of ruin indicated.
Multi-Threshold MAP Risk Model

- Thresholds: \( 0 = b_0 < b_1 < \cdots < b_n < b_{n+1} = \infty \)
- Premium rate \( c_k \) for \( b_{k-1} \leq u < b_k, \ k = 1, \cdots, n + 1 \)
  \[ c = c_1 > c_2 > \cdots > c_n > c_{n+1} \geq 0 \]
- Time of ruin \( \tau_B = \inf\{t \geq 0 : U_B(t) < 0\} \)
- Surplus process \( \{U_B(t); t \geq 0\} \) satisfies
  \[ dU_B(t) = c_k dt - dS(t), \quad b_{k-1} \leq U_B(t) < b_k \]
- Claim amounts distribution \( f_{i,j}, F_{i,j} \) and Laplace transformation \( \hat{f}_{i,j}(s) \)
Expected Discounted Dividend Payments

- $D(t)$ is the aggregate dividends paid by time $t$
- Define
  \[ D_{u,B} = \int_0^{\tau_B} e^{-\delta t} dD(t), \quad u \geq 0, \]
  to be the present value of dividend payments prior to ruin, given the initial surplus $u$
- Define
  \[ V_i(u; B) = \mathbb{E}_i[D_{u,B} \mid U_B(0) = u], \quad i \in E, \]
  to be the expected present value of dividend payments prior to ruin, given the initial surplus $u$ and the initial phase $i \in E$
The piecewise vector function of the expected present value of the total dividend payments prior to ruin

\[
\vec{V}(u; B) = \begin{cases} 
\vec{V}_1(u; B) & 0 \leq u < b_1, \\
\vec{V}_k(u; B) & b_{k-1} \leq u < b_k, \quad k = 2, \cdots, n, \\
\vec{V}_{n+1}(u; B) & b_n \leq u < \infty.
\end{cases}
\]

\[
\vec{V}_k(u; B) = (V_{1,k}(u; B), \cdots, V_{m,k}(u; B))^\top
\]
for \( b_{k-1} \leq u < b_k \) and \( k = 1, \cdots, n + 1 \).
Differential Approach

- Typical approach in various risk models
- Integro-differential equations are involved
- Can be derived and solved analytically for some families of claim amounts distribution
- Mainly in Gerber-Shiu discounted penalty function
  Techniques can be applied to the dividend payments problems
- Lin and Sendova (2008), classical risk model
- Lu and Li (2009), Sparre Andersen risk model
Integro-Differential Equation for $\vec{V}_k(u; B)$

- Condition on the events occurring in a small time interval $[0, h]$
  - No change in the MAP state
  - A change in the MAP state accompanied by no claim arrival
  - A change in the MAP state accompanied by a claim arrival; Claim amounts may vary
  - Two or more events occur
Integro-Differential Equation for $\vec{V}_k(u; B)$

- Integro-differential equation, for $b_{k-1} \leq u < b_k$

$$c_k \vec{V}_k'(u; B) = \delta \vec{V}_k(u; B) - D_0 \vec{V}_k(u; B) - \int_0^{u-b_{k-1}} \Lambda_f(x) \vec{V}_k(u-x; B) dx - \vec{\gamma}_k(u)$$

where $\gamma_{i,k}(u) = (c - c_k) + \sum_{j=1}^{m} D_1(i,j) \sum_{l=1}^{k-1} \int_{u-b_l}^{u-b_{l-1}} V_{j,l}(u-x; B) dF_{i,j}(x)$

- Solution

$$\vec{V}_k(u; B) = \vec{v}_k(u - b_{k-1}) \vec{V}_k(b_{k-1}; B) - \frac{1}{c_k} \int_0^{u-b_{k-1}} \vec{v}_k(t) \vec{\gamma}_k(u - t) dt$$

where $\vec{v}_k(u - b_{k-1}) = \mathcal{L}^{-1} \left\{ \left( s - \frac{\delta}{c_k} \right) I + \frac{1}{c_k} (D_0 + \Lambda_f(s)) \right\}^{-1}$
Recursive Expression for $\vec{V}_k(u; B)$

- Define vector function $\vec{V}_k(u)$ for $u \geq b_{k-1}$

$$\vec{V}_k(u) = v_k(u - b_{k-1})\vec{V}_k(b_{k-1}) - \frac{1}{c_k} \int_0^{u-b_{k-1}} v_k(t)\vec{\gamma}_k(u - t)dt$$

- Restrict to $b_{k-1} \leq u < b_k$, compare with $\vec{V}_k(u; B)$

$$\vec{V}_k(u; B) = \vec{V}_k(u) + v_k(u - b_{k-1})\vec{\pi}_k(B), \quad b_{k-1} \leq u < b_k$$

- Continuity condition at $b_{k-1}$, $k = 1, \cdots, n$

$$\vec{\pi}_{k+1}(B) = \vec{V}_k(b_k) - \vec{V}_{k+1}(b_k) + v_k(b_k - b_{k-1})\vec{\pi}_k(B)$$

- Final boundary condition when $k = n + 1$

$$\vec{\pi}_{n+1}(B) = \vec{V}_n(b_n) - \vec{V}_{n+1}(b_n) + v_n(b_n - b_{n-1})\vec{\pi}_n(B) = \vec{0}$$
Layer-Based Recursive Algorithm

- Computational disadvantage of the recursive algorithm based on integro-differential equations
  - Constant vectors can only be solved in the last layer
  - Infeasible to compute for large number of layers

- Layer-based approach
  - Condition on the exit times of the surplus out of each layer
  - Calculate successively for increasing number of layers

\[
\text{The } k\text{-layer model } \Leftarrow \begin{cases} 
\text{The } (k-1)\text{-layer model} \\
\text{Classical one-layer model}
\end{cases}
\]

Reference: Albrecher and Hartinger (2007)
Sample Path of One-Layer Model with Dividend Payments
Time Value of Upper Exit

- Define $\tau^*(u, a, b) = \inf\{ t \geq 0 : U(t) \notin [a, b]|U(0) = u\}$
- Define
  
  $$\tau^+(u, a, b) = \begin{cases} 
  \tau^*(u, a, b) & \text{if } U(\tau^*(u, a, b)) = b \\
  \infty & \text{if } U(\tau^*(u, a, b)) < a 
  \end{cases}$$

  and

  $$\tau^-(u, a, b) = \begin{cases} 
  \infty & \text{if } U(\tau^*(u, a, b)) = b \\
  \tau^*(u, a, b) & \text{if } U(\tau^*(u, a, b)) < a 
  \end{cases}$$

- Laplace transform of $\tau_k^+(u, 0, b)$

  $$B_{i,j,k}(u, b) = \mathbb{E} \left[ e^{-\delta \tau_k^+(u, 0, b)} \mathbf{1}_{[J(\tau_k^+(u, 0, b))=j]} | J(0) = i \right]$$

  given initial phase $i$ and reaching $b$ in phase $j$

Time Value of Upper Exit

For $\delta > 0$ and $k \in \mathbb{N}^+$, we have

1. $B_k = 1$, if $u \geq b$
2. $B_k = 0$, if $u < 0$
3. For $0 \leq u < b_{k-1}$
   
   $$B_k(u, b) = \begin{cases} 
   B_{k-1}(u, b), & \text{if } b \leq b_{k-1} \\
   B_{k-1}(u, b_{k-1})B_k(b_{k-1}, b), & \text{if } b \geq b_{k-1} 
   \end{cases}$$

4. For $b_{k-1} \leq u \leq b$
   
   $$B_k(u, b) = B_{1,k}(u - b_{k-1}, b - b_{k-1}) + M_k(u - b_{k-1}) - B_{1,k}(u - b_{k-1}, b - b_{k-1})M_k(b - b_{k-1})$$

- Parallel results in matrix form

Reference: Albrecher and Hartinger (2007)
Sample Path for $0 \leq u \leq b_{k-1}$

Surplus $U_B(t)$

\[
\begin{align*}
\text{after hitting } b_{k-1}: & \quad \sum_{l=1}^{m} B_{l,l-1}(u,b_{k-1}) V_{l,k}(b_{k-1};B) \\
\text{before hitting } b_{k-1}: & \quad V_{l,k-1}(u;B) - \sum_{l=1}^{m} B_{l,l-1}(u,b_{k-1}) V_{l,k-1}(b_{k-1};B)
\end{align*}
\]
Sample Path for $u \geq b_{k-1}$

$$U_{1,k}(t)$$

**Case $0 \leq u < b_{k-1}$**

$$U_B(\tau_k^-(u,b_{k-1},b)) = b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|$$

Surplus $U_0(t)$
Expected Discounted Dividend Payments

- For $0 \leq u \leq b_{k-1}$
  \[ \tilde{V}_k(u; B) = \tilde{V}_{k-1}(u; B) + B_{k-1}(u, b_{k-1}) \left[ \tilde{V}_k(b_{k-1}; B) - \tilde{V}_{k-1}(b_{k-1}; B) \right] \]

- For $u \geq b_{k-1}$
  \[
  \tilde{V}_k(u; B) = \tilde{V}_{1,k}(u - b_{k-1}) + \mathbb{E} \left[ e^{-\delta \tau_{1,k}(u-b_{k-1})} \tilde{V}_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(u-b_{k-1}))|; B) \right]
  \]
“Contagion” Example

- State A: standard claims, $\lambda_1 = 1$, $1/\beta_1 = 1/5$
- State B: additional infectious claims, $\lambda_2 = 10$, $1/\beta_2 = 3$
- State A $\rightarrow$ B, $\alpha_A = 0.02$; State B $\rightarrow$ A, $\alpha_B = 1$
- $D_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{pmatrix}$, $D_0 = \begin{pmatrix} -\alpha_A - \lambda_1 & \alpha_A \\ \alpha_B & -\alpha_B - \lambda_1 - \lambda_2 \end{pmatrix}$
- Thresholds $(0, 20, 40, \infty)$, premium rates $(2, 1.5, 1)$

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<th>$u$</th>
<th>$\delta = 0.1$</th>
<th>$\delta = 0.01$</th>
<th>$\delta = 0.001$</th>
<th>Badescu et al. (2007)</th>
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Conclusion

- Differential approach is applicable to the MAP risk model
- Moment generating function and higher moments
- Layer-based approach provides an alternative method
Reference