BETA-GAMMA ALGEBRA, DISCOUNTED CASH-FLOWS, AND BARNES’ LEMMAS

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Discounted cash-flows

Suppose i.i.d. cash-flows \( \{C_n\} \) occur at times \( \{T_n\} \), and that one wishes to find the distribution of the discounted value of all future cash-flows. If the discount rate is \( r > 0 \), then this is

\[
X = \sum_{n=1}^{\infty} e^{-rT_n} C_n.
\]

Let the waiting times

\[
W_1 = T_1, \quad W_n = T_n - T_{n-1}, \quad n \geq 2,
\]

be i.i.d., making \( \{T_n\} \) a renewal process, and assume moreover that \( \{T_n\} \) and \( \{C_n\} \) are independent. Then the above sum may be rewritten
as
\[ X = \sum_{n=1}^{\infty} A_1 \cdots A_n C_n \quad \text{if} \quad A_n = e^{-rW_n}. \]

Such sums of products of random variables occur in a variety of applications and have been studied for several decades.

It is known that in such cases \( X \) satisfies the identity in law
\[ X \overset{d}{=} A(X + C). \]

A known example is:
\[ G_1^{(a)} \overset{d}{=} B^{(a,b)}\left(G_1^{(a)} + G_2^{(b)}\right), \]

where all variables on the right are independent and
\[ B^{(a,b)} \sim \text{Beta}(a, b), \quad G_1^{(a)} \sim \text{Gamma}(a, 1), \quad G_2^{(b)} \sim \text{Gamma}(b, 1). \]
This means that

\[
\sum_{n=1}^{\infty} B_1^{(a,b)} \cdots B_n^{(a,b)} G_n^{(a)} \sim \text{Gamma}(a, 1)
\]

(all variables independent).

The identity

\[
G_1^{(a)} \overset{d}{=} B^{(a,b)}(G_1^{(a)} + G_2^{(b)}),
\]

is the same as

\[
G_1^{(a)} \overset{d}{=} B^{(a,b)} G_2^{(a+b)}
\]

This is part of the so-called “beta-gamma algebra”. It may be proved with Mellin transforms, i.e. by checking that

\[
E[G_1^{(a)}]^p = E[B^{(a,b)} G_2^{(a+b)}]^p, \quad p \geq 0.
\]
Mellin transform: if $X \geq 0$, then $\mathcal{M}_X(s) = \mathbb{E}X^s$.

Mellin transforms for sums of positive variables

**Theorem A.** Suppose $c > 0$, $\text{Re}(p) > c$ and

$$\mathbb{E}(X_1^{-c}X_2^{c-\text{Re}(p)}) < \infty.$$  

Then

$$\mathbb{E}(X_1 + X_2)^{-p} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \mathbb{E}(X_1^{-z}X_2^{z-p}) \frac{\Gamma(z) \Gamma(p-z)}{\Gamma(p)}.$$  


Barnes’ Lemmas and properties of beta and gamma variables

Barnes’ First Lemma (Barnes, 1908). For a suitably curved line of integration, so that the decreasing sequences of poles lie to the left and the increasing sequences lie to the right of the contour,

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz \Gamma(A + z)\Gamma(B + z)\Gamma(C - z)\Gamma(D - z) = \frac{\Gamma(A + C)\Gamma(A + D)\Gamma(B + C)\Gamma(B + D)}{\Gamma(A + B + C + D)}.
\]

Theorem B. By Theorem A, Barnes’ First Lemma is equivalent to the additivity property of gamma distributions: if \(a, b > 0\) and \(G_1^{(a)}, G_2^{(b)}\) are independent, then \(G_1^{(a)} + G_2^{(b)} \overset{d}{=} G_3^{(a+b)}\).
Barnes’ Second Lemma (Barnes, 1910). For a suitably curved line of integration, so that the decreasing sequences of poles lie to the left and the increasing sequences lie to the right of the contour, if \( E = A + B + C + D \),

\[
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dz \frac{\Gamma(A + z)\Gamma(B + z)\Gamma(C + z)\Gamma(D - z)\Gamma(-z)}{\Gamma(E + z)} = \frac{\Gamma(A)\Gamma(B)\Gamma(C)\Gamma(A + D)\Gamma(B + D)\Gamma(C + D)}{\Gamma(E - A)\Gamma(E - B)\Gamma(E - C)}.
\]
Another properties of gamma variables (Dufresne, 1998)

Proposition C. Suppose all variables are independent.

For any $a, b, c > 0$,

$$B_1^{(a,b+c)} G_1^{(b)} + G_2^{(c)} \overset{d}{=} G_3^{(b+c)} B_2^{(a+c,b)} \overset{d}{=} G_4^{(a+c)} B_3^{(b+c,a)}.$$ 

Theorem D. By Theorem A, Barnes’ Second Lemma is equivalent to the property in Proposition C.
Properties of reciprocal gamma variables

We look at the distribution of

\[ H^{(a,b)} = \left( \frac{1}{G_1^{(a)}} + \frac{1}{G_2^{(b)}} \right)^{-1} = \frac{G_1^{(a)} G_2^{(b)}}{G_1^{(a)} + G_2^{(b)}}, \]

where \( a, b > 0 \) and the the gamma variables are independent.

The distribution of \( H^{(a,b)} \) turns out to be directly related to the “beta product distribution”.

**Proposition E.** The distribution of the product of independent \( B^{(a,b)} \) and \( B^{(c,d)} \) extends to a four-parameter family called the “beta product” distribution. It is a proper probability distribution on \((0, 1)\) if, and only if, the parameters \((a, b, c, d)\) satisfy:
a, c, b + d, \text{Re}(a + b), \text{Re}(c + d) > 0, \text{ and either }

(i) (real case) b, d are real and \( \min(a, c) < \min(a + b, c + d) \), or

(ii) (complex case) \( \text{Im}(b) = -\text{Im}(d) \neq 0 \) and \( a + b = c + d \).

“\( B^{(a, b, c, d)} \)” will represent a variable with that distribution.

The density of \( B^{(a, b, c, d)} \) is

\[
\frac{\Gamma(a + b)\Gamma(c + d)}{\Gamma(a)\Gamma(c)\Gamma(b + d)} u^{a-1}(1-u)^{b+d-1} \ _2F_1(a+b-c, d; b+d; 1-u)1_{\{0<u<1\}}.
\]
Theorem F. (a) If \( \text{Re}(p) > -\min(a, b) \), then

\[
\mathbb{E} (H^{(a,b)})^p = \frac{(a)_p(b)_p(a + b)_p}{(a + b)_{2p}}.
\]

(b) For any \( 0 < a, b < \infty \),

\[
H^{(a,b)} = \frac{G_1^{(a)}G_2^{(b)}}{G_1^{(a)} + G_2^{(b)}} \xrightarrow{d} \frac{1}{4} B(a, \frac{b-a}{2}, b, \frac{a-b+1}{2}) G(a+b),
\]

where the variables on the right are independent. This is the same as:

\[
\frac{1}{G^{(a)}} + \frac{1}{G^{(b)}} \xrightarrow{d} \frac{4}{B(a, \frac{b-a}{2}, b, \frac{a-b+1}{2})} \cdot \frac{1}{G^{(a)} + G^{(b)}}
\]
(c) For $a, b > 0$ and $\text{Re}(s) > -4$,

$$
\mathbb{E}e^{-sH^{(a,b)}} = {3F}_2(a, b, a + b; \frac{a+b}{2}, \frac{a+b+1}{2}; -\frac{s}{4}).
$$

(d) For any $a, b > 0$,

$$(G_1^{(a+b)})^2 H^{(a,b)} \overset{\text{d}}{=} \frac{G_2^{(a)} G_3^{(b)} G_4^{(a+b)}}{G_1^{(a)}},$$

where the variables on either side are independent.

**Corollary G.** (a) The identity in law

$$
\frac{1}{G_1^{(a)}} \overset{\text{d}}{=} A \left( \frac{1}{G_2^{(a)}} + \frac{1}{G_3^{(b)}} \right),
$$
with independent variables on the right, has a solution $A$ if, and only if, one of the three cases below occurs:

(i) $0 < a < b < \infty$, $b > \frac{1}{2}$. Then

$$A \overset{d}{=} \frac{1}{4B\left(\frac{a+b}{2}, \frac{b-a}{2}, \frac{a+b+1}{2}, \frac{a+b-1}{2}\right)}.$$  

(ii) $a = b > \frac{1}{2}$. Then

$$A \overset{d}{=} \frac{1}{4B\left(a+\frac{1}{2}, a-\frac{1}{2}\right)}.$$  

(iii) $a = b = \frac{1}{2}$. Then $A = \frac{1}{4}$ and

$$\frac{4}{G_1^{(\frac{1}{2})}} \overset{d}{=} \frac{1}{G_2^{(\frac{1}{2})}} + \frac{1}{G_3^{(\frac{1}{2})}}.$$
(b) In any one of the three cases above, let

\[ A_n = \frac{1}{4B_n^{(a+b, b-a, \frac{a+b+1}{2}, \frac{a+b-1}{2})}}, \quad n = 1, 2, \ldots \]

Then, if all variables are independent,

\[ \sum_{n=0}^{\infty} A_1 \cdots A_n \frac{1}{G_n^{(b)}} \overset{d}{=} \frac{1}{G_0^{(a)}}. \]

References


